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STATISTICAL SYSTEMS OF PARTICLES IN THE EXPANDING UNIVERSE

By C. GILBERT (*Newcastle upon Tyne*)

[Received 26 May 1950; in revised form 10 August 1951]

1. Introduction

In Milne's theory of world structure† there exists a system of fundamental particles. Attached to each particle is a privileged observer, and each observer describes the whole system in the same way in terms of his own coordinates. The system is then said to *satisfy the cosmological principle*. This system of observers and fundamental particles has been called by Milne the *substratum*. Relative to the observers of this system Milne also defines a statistical system of particles, described in the same way by each of the observers in terms of his own coordinates. An essential feature of the theory is that measurements of the coordinates of distant events are made by each observer operationally with a clock and theodolite using a light-signalling procedure.

H. P. Robertson‡ has shown that the operational method of Milne can be used to describe more general kinematic and statistical systems satisfying the cosmological principle. He has shown that there exists a Riemannian space-time for the system of observers, having an invariant quadratic metric

$$ds^2 = d\tau^2 - \frac{R^2(\tau)}{c^2}\, du^2, \tag{1}$$

where

$$du^2 = h_{\alpha\beta}\, dx^\alpha dx^\beta$$

is a positive definite 3-space of constant curvature $k = +1, 0$, or -1.§ The world-lines of observers are the geodesics $x^\alpha =$ constant, and the light paths are the null geodesics of (1); c is a constant which has the same value for every observer, and $R(\tau)$ is an arbitrary function of τ. The substratum proposed by Milne corresponds to the case when the metric (1) has $R(\tau) = c\tau$, $k = -1$.‖

In Milne's theory the system of fundamental particles is treated as being distinct from the statistical system of particles. The latter system

† *Relativity, Gravitation, and World Structure* (Oxford, 1935).

‡ *Astrophys. J.* 82 (1935), 284; 83 (1936), 187, 257.

§ Greek suffixes will take the values 1, 2, 3 and Roman suffixes the values 0, 1, 2, 3; x^0 will be written for the time coordinate τ, and the summation convention will apply for a repeated suffix.

‖ In this case τ corresponds to Milne's kinematic time and ζ, defined by equation (3), corresponds to Milne's $\xi^{\frac{1}{2}}$.

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is superposed on the substratum, but also the substratum can be considered free from a statistical system. The problem Milne tried to solve was: *What is the motion of a free test-particle which is projected in a substratum alone?* Milne had considerable difficulty in solving this problem, and the main arguments by which he deduced the path of a free test-particle for a substratum alone depended on (*a*) the construction of a dynamics for the substratum, which approximates to the Newton–Einstein dynamics† and (*b*) the deduction of an inverse square law of gravitation for an isolated condensation of mass.‡ In neither of these arguments does it appear that the mass of a fundamental particle is, in itself, of real significance. In (*a*) the substratum is essentially considered to be replaced by a continuous distribution of mass having a density represented by a continuous function of the coordinates. In (*b*), on the other hand, a microscopic analysis is carried out in which the fundamental particles are widely separated and have superposed on them a statistical system. In order to get a sensible result from this analysis it is necessary to assume that the mass of each fundamental particle is negligibly small compared with the mass of the condensation of statistical particles which is superposed on it.

I do not question the correctness of the equations giving the path of a free test-particle in Milne's substratum, in the case when the substratum is taken to be a continuous distribution of mass, but in my opinion there is no necessity to introduce both the fundamental particles and the statistical particles into kinematic theory. Milne's system of observers is defined kinematically, and it is not necessary to assume that they are attached to material particles if each observer is only used to determine a frame of reference relative to which the universe is described. I shall assume that all the material energy of the universe§ is contained in the statistical system which is superposed on the system of observers. These observers I call the *fundamental* observers.

In this paper a *kinematic system* is defined to be a system of fundamental observers, together with a certain statistical system of particles. These systems are defined with respect to the metric (1). The paths of the particles of the statistical system are given by equations of a particular form,‖ which are shown to lead to the result that the energy of each

† *Proc. Royal Soc.* A, 154 (1936), 22.

‡ Ibid. A, 156 (1936), 62.

§ The energy of the radiation field is not considered in this paper.

‖ The procedure is equivalent to setting $m = 0$, $c = 0$ in Milne, *Kinematic Relativity* (Oxford, 1948) Ch. IX, so that both the inertial mass and gravitational mass of the fundamental particles are zero for a kinematic system.

particle always has a constant value. The particles I call the *free particles*, and their paths the *free paths* of the kinematic system.

I calculate the material-energy tensor for the kinematic system. This tensor has the form of the energy tensor of a perfect fluid, each fundamental observer being at rest relative to the fluid in his immediate neighbourhood. I define a *kinematic model* M of the universe to be the limiting case of a kinematic system when the pressure $p \to 0$ and $\rho \neq 0$. In the case $R(\tau) = c\tau, k = -1$ the kinematic model M has the properties of Milne's substratum, any free path of M corresponding to the path of a free test-particle in Milne's theory.

An important property possessed by a kinematic system is that it can be transformed, by regraduating the scales of local coordinates, into a kinematic system which is defined with respect to a different metric. In particular this property holds for the model M. I use the theory of regraduation to show that a model of class $M(R, \tau, k)$† can be transformed into a model of class $\bar{M}(\bar{R}, \bar{\tau}, k)$ which is defined with respect to the metric

$$d\bar{s}^2 = d\bar{\tau}^2 - \frac{\bar{R}^2(\bar{\tau})}{c^2}\, du^2, \tag{2}$$

where $\bar{s}$, $\bar{\tau}$ are new measures of proper time and coordinate time respectively, and $\bar{R}(\bar{\tau})$ is a function of $\bar{\tau}$. In § 6, I use this property of the model M to show its relation to certain other cosmological models.

2. The statistical system

The metric (1) defines a space-time for the system of fundamental observers. These observers have a number density $\propto R^{-3}(\tau)$. Superposed on this system of observers there is a statistical system of particles, each particle having the same proper mass m. This statistical system is described in the same way by each observer in terms of his own coordinates.

At an event E having coordinates x^i, let δ^i_0 be a vector having the components $(1, 0, 0, 0)$, which is tangent to the world line of a fundamental observer, and let λ^i be a unit vector which is tangent to the world-line of a material particle. Then an invariant quantity ζ is defined by

$$\zeta = g_{ij}\lambda^i\delta^j_0, \tag{3}$$

where g_{ij} is the fundamental tensor of the metric (1).

Also the number of particles dN within an invariant volume $d\Sigma$ of

† The symbols in parentheses refer to the metric of space-time. R is written for the function $R(\tau)$, τ is the coordinate time, and k is the curvature of the u-space.

3-space, corresponding to the range of values x^α, $x^\alpha+dx^\alpha$, and having velocity 4-vectors in the 3-dimensional solid angle $d\Omega$, corresponding to the range of values λ^α, $\lambda^\alpha+d\lambda^\alpha$, is†

$$dN = \frac{\Psi(\tau,\zeta)}{R^3(\tau)}\,d\Sigma d\Omega. \tag{4}$$

$\Psi(\tau,\zeta)$ is a statistical distribution function which, in general, is a function of τ and ζ, and $d\Sigma$, $d\Omega$ are given by

$$d\Sigma = \zeta R^3 h^{\frac{1}{2}}\,dx^1dx^2dx^3, \qquad d\Omega = c^{-3}\zeta^{-1}R^3h^{\frac{1}{2}}\,d\lambda^1d\lambda^2d\lambda^3. \tag{5}$$

The equations of motion of the material particles can be shown to have the form

$$\frac{d\lambda^i}{ds}+\{jk,i\}\lambda^j\lambda^k = \Gamma(\tau,\zeta)(\delta^i_0-\zeta\lambda^i), \tag{6}$$

where $\{jk,i\}$ are the Christoffel symbols formed with respect to g_{ij}. $\Gamma(\tau,\zeta)$ is a function of τ and ζ which must satisfy the generalized form of the Boltzmann equation

$$\frac{\partial}{\partial\tau}[\zeta(\zeta^2-1)^{\frac{1}{2}}\Psi] = \frac{\partial}{\partial\zeta}\left[(\zeta^2-1)^{\frac{1}{2}}\left(\Gamma+\frac{1}{R}\frac{dR}{d\tau}\right)\Psi\right]. \tag{7}$$

This equation is obtained on the assumption that the effect of collisions between particles can be neglected.

3. The kinematic system

We define a kinematic system to be a system of fundamental observers for whom there exists the invariant quadratic metric (1), together with a statistical system of particles, whose paths are given by equations (6) with

$$\Gamma = -\frac{1}{R}\frac{dR}{d\tau}. \tag{8}$$

In making this choice of Γ we have been guided by the result obtained by Milne for his substratum, that the energy of a free particle remains constant during its motion. This is proved below for our kinematic system, but, as has been explained in the introduction, our fundamental observers are without mass, and hence Milne's problem of *deriving* the equations of motion of a free particle in the presence of the fundamental particles alone does not arise.

Using (8) we find that equations (6) can then be written in the form

$$\zeta = K, \qquad \frac{d}{d\tau}\left(R\frac{dx^\alpha}{d\tau}\right)+\{\beta\gamma,\alpha\}R\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0, \tag{9}$$

† Equations (4), (5), (6), and (7) below, correspond to the equations (7.3), (7.2), (4.4), and (7.9) respectively, in Robertson's papers (loc. cit. 1936) with some slight changes in form.

where K is an arbitrary constant, and $\{\beta\gamma, \alpha\}$ are the Christoffel symbols formed with respect to $h_{\alpha\beta}$. The particles forming the statistical system we call the *free particles* and their paths the *free paths of the kinematic system*. The equations of the light paths, or null geodesics of V_4, are also included in the form (9) in the case $K = \infty$.

From equations (7) and (8) we find that

$$\Psi(\tau, \zeta) = \psi(\zeta), \tag{10}$$

for a kinematic system, where $\psi(\zeta)$ is an arbitrary function of ζ.

Through an event E of V_4 there passes the world-line of one fundamental observer, and we shall find it convenient to describe the infinitesimal neighbourhood of E in terms of a *local coordinate system* which is defined relative to that observer in the following manner. In terms of local coordinates dX^i, space-time has the metric

$$ds^2 = \delta_{ij}\, dX^i dX^j, \tag{11}$$

where the components of δ_{ij} have the Galilean values

$$\delta_{00} = 1, \qquad \delta_{\alpha\alpha} = -1/c^2, \qquad \delta_{\alpha\beta} = 0 \quad (\alpha \neq \beta).$$

The event E has coordinates $dX^i = 0$, and events on the world-line of the fundamental observer have $dX^\alpha = 0$. These local coordinates are related to the coordinates x^i by equations of the form

$$dX^i = A_l^{(i)}(x)\, dx^l, \tag{12}$$

where the $A_l^{(i)}$ are four covariant tensors ($i = 0, 1, 2, 3$) which are in general functions of x^i such that $|A_l^{(i)}| \neq 0$. From the way in which the local coordinates have been defined we have

$$\delta_{ij} A_l^{(i)} A_m^{(j)} = g_{lm}, \tag{13}$$

$$A_0^{(0)} = 1, \qquad A_\alpha^{(0)} = A_0^{(\alpha)} = 0. \tag{14}$$

Since the light paths are null curves of (1), they will also be null curves of (11) and therefore c is the velocity of light in local coordinates. Corresponding to a vector λ^i which is tangent to a free path at E, we have a unit tangent vector Λ^i in local coordinates given by $\Lambda^i = A_l^{(i)} \lambda^l$. The momentum vector associated with the particle is $mc\Lambda^i$, and according to the special theory of relativity the energy $\mathscr{E}$ of the particle relative to the fundamental observer is

$$\mathscr{E} = mc^2 \delta_{ij} \Lambda^i \delta_0^j.$$

From (3) and (14) we then find $\mathscr{E} = mc^2\zeta$ and therefore:

The energy of a free particle is given by the invariant $mc^2\zeta$ which is constant along a free path.

4. The material-energy tensor for the kinematic system

The proper number density of particles at an event E whose world-lines have unit tangent vectors in the range λ^α, $\lambda^\alpha+d\lambda^\alpha$ is $dN/d\Sigma$. Hence from (4), (5), and (10), the material-energy tensor for the substratum is

$$T^{ij} = m\int \frac{\psi(\zeta)}{R^3}\lambda^i\lambda^j\,d\Omega = mc^{-3}\int\int\int \psi(\zeta)\zeta^{-1}\lambda^i\lambda^j h^{\frac{1}{2}}\,d\lambda^1 d\lambda^2 d\lambda^3, \tag{15}$$

where it is assumed that each particle has the same proper mass m. These integrals can be evaluated by introducing normal coordinates for the 3-space du^2. Writing $R^2h_{\alpha\beta}\lambda^\alpha\lambda^\beta c^{-2} = z^2$, we have $\zeta = \sqrt{(1+z^2)}$, and $\psi(\zeta) = \chi(z)$, where $\chi(z)$ is a function of z only. We thus find the following non-vanishing components of T^{ij},

$$T^{00} = A/R^3, \qquad T^{\alpha\beta} = -Bg^{\alpha\beta}/R^3,$$

where

$$A = 4\pi m\int_0^\infty \chi(z)(1+z^2)^{\frac{1}{2}}z^2\,dz, \qquad B = \tfrac{4}{3}\pi m\int_0^\infty \chi(z)(1+z^2)^{-\frac{1}{2}}z^4\,dz. \tag{16}$$

Hence
$$T^{ij} = (\rho+p/c^2)\delta^i_0\,\delta^j_0 - p/c^2\,g^{ij}, \tag{17}$$

where
$$\rho = A/R^3, \qquad p/c^2 = B/R^3. \tag{18}$$

Since δ^i_0 is the proper vector of the observer at E, we see from the form of (17) that T^{ij} is the energy-tensor of a perfect fluid of proper density ρ and pressure p referred to proper coordinates.

It is usual to assume in constructing models of the universe that the pressure of matter is either zero, or extremely small. From the integrals for A and B given in (16) we can show, by choosing a suitable form for $\chi(z)$, that p can be made less than any assignable quantity however small, whilst $\rho \neq 0$.†

In the limit $p \to 0$, $\rho \neq 0$ I shall say that we have *a kinematic model of the universe of class* $M(R, \tau, k)$. In this limiting case $\psi(\zeta) \to 0$ for all ζ except $\zeta = 1$, and the free particles therefore coincide with the fundamental observers. The free paths of a kinematic system given by (9) are independent of the statistical distribution function, and in this limiting case can be considered to be the paths of free particles whose number density is so small that they have a negligible effect on the model M. The concept of a free path in this case is the same as the concept of a 'free test particle' in Milne's theory, and also in the general theory of relativity.

† We can take, for example, $\chi(z) = a^{-1}bz^{-2}(1+z^2)^{\frac{1}{2}}\exp(-z/a)$, where a and b are constants. Then we find $A = 4\pi mb(1+2a^2)$, $B = \frac{8}{3}\pi ma^2b$, from which the result follows on taking a indefinitely small, and $mb \neq 0$.

5. Regraduation of scales

A regraduation consists of a change of the scales of the local coordinates of the form

$$d\bar{X}^i = e^{\sigma(\tau)}\, dX^i, \tag{19}$$

where $\sigma(\tau)$ is a function of τ only. This change of the scales is made at each event of V_4, the coordinates x^i remaining unchanged. At each event E we have a new local space-time having the metric

$$d\bar{s}^2 = \delta_{ij}\, d\bar{X}^i d\bar{X}^j, \tag{20}$$

and from (12), (13), (19), and (20) we find that V_4 is transformed into a conformal space-time $\overline{V}_4$ having the metric

$$d\bar{s}^2 = e^{2\sigma(\tau)} g_{ij}\, dx^i dx^j. \tag{21}$$

By making a transformation of the time coordinate

$$d\bar{\tau} = e^{\sigma(\tau)}\, d\tau, \tag{22}$$

the metric of $\overline{V}_4$ can be written in the alternative form (2) where

$$\bar{R}(\bar{\tau}) = e^{\sigma(\tau)} R(\tau). \tag{23}$$

A regraduation in the sense defined above is in agreement as a special case with the definition of a regraduation given by A. G. Walker.†

I shall now prove the theorem that:

Any model of class $M(R, \tau, k)$ can be transformed into a model of class $\overline{M}(\bar{R}, \bar{\tau}, k)$ by a regraduation and a transformation of the time coordinate. I shall say that M and $\overline{M}$ are *equivalent* models. Since k is unchanged by the regraduation, we can assume that the space coordinates x^α are also unchanged. The world-lines of fundamental observers will then be given by $x^\alpha =$ constant, both before and after the transformation.

Let quantities $\bar{\zeta}, \bar{\lambda}^i, \bar{\psi}(\bar{\zeta}), d\bar{N}, d\bar{\Sigma}, d\bar{\Omega}, \bar{T}^{ij}, \bar{\rho}, \bar{p}$ be defined in $\overline{V}_4$, having the metric (2), in the same way as the corresponding quantities have been defined in V_4. Then for corresponding events in $\overline{V}_4$ and V_4 we find, by the use of (21) and (22), that

$$\bar{\zeta} = \zeta, \qquad \bar{\lambda}^\alpha = e^{-\sigma(\tau)}\lambda^\alpha, \qquad d\bar{\Sigma} = e^{3\sigma(\tau)}\, d\Sigma, \qquad d\bar{\Omega} = d\Omega. \tag{24}$$

From (22), (23), and the first of equations (24) we find that equations (9) transform into

$$\bar{\zeta} = K, \qquad \frac{d}{d\bar{\tau}}\left(\bar{R}\,\frac{dx^\alpha}{d\bar{\tau}}\right) + \{\beta\gamma, \alpha\}\bar{R}\,\frac{dx^\beta}{d\bar{\tau}}\,\frac{dx^\gamma}{d\bar{\tau}} = 0. \tag{25}$$

The light paths and the free paths of M therefore transform into the light paths and free paths respectively of $\overline{M}$.

† *Proc. Roy. Soc. Edinburgh* (1946), 164.

The same physical system will be described in $\overline{V}_4$ as in V_4, if $d\bar{N} = dN$. Using (24) we thus find that

$$\bar{\psi}(\bar{\zeta}) = \psi(\zeta)$$

and then, from (15) and the corresponding equations for $\overline{V}_4$, we find

$$\overline{T}^{00} = e^{-3\sigma(\tau)}T^{00}, \qquad \overline{T}^{\alpha\beta} = e^{-5\sigma(\tau)}T^{\alpha\beta}.$$

It follows that a model M having $\rho = A/R^3(\tau)$, $p = 0$ will transform into a model $\bar{M}$ having $\bar{\rho} = A/\bar{R}^3(\bar{\tau})$, $\bar{p} = 0$, where A is defined by the first of equations (16). For any functions $R(\tau)$, $\bar{R}(\bar{\tau})$ a regraduation defined by $\sigma(\tau)$ can be determined from equations (22) and (23), and hence the theorem stated above is proved.

6. Equivalence of the kinematic models to other cosmological models

(*a*) *The Lemaître models.* The energy tensor T^{ij} for these models must satisfy Einstein's field equations. When the V_4 has the metric (1), the density ρ and pressure p of matter satisfy the equations†

$$\kappa\rho = 3R^{-2}\left[\left(\frac{dR}{d\tau}\right)^2 + c^2k\right] - \lambda, \tag{26}$$

$$\kappa p/c^2 = -R^{-2}\left[2R\frac{d^2R}{d\tau^2} + \left(\frac{dR}{d\tau}\right)^2 + c^2k\right] + \lambda, \tag{27}$$

where κ is Einstein's constant, and λ is the cosmical constant.

Setting $\rho = A/R^3(\tau)$, $p = 0$, which are the values for the model M, we find from (26) and (27) that $R(\tau)$ and k must satisfy the equation

$$2R^2\frac{d^2R}{d\tau^2} - 2R\left[\left(\frac{dR}{d\tau}\right)^2 + c^2k\right] + \kappa A = 0. \tag{28}$$

A first integral of this equation gives (26), where λ is the constant of integration.

It has been pointed out by Walker‡ that, when the conservation equations $T^{ij}_{,j} = 0$ are based on the Principle of Equivalence, the path of a free test-particle is not necessarily a geodesic of space-time. We shall assume that these paths for the Lemaître models are given by equations (9) for the free paths of the model M. Equation (28) then determines Lemaître models having the same properties as the kinematic models. From the theory of regraduation we then have the result that:

The kinematic models of class $M(R, \tau, k)$ are equivalent to the Lemaître models of pressure $p = 0$.

† H. P. Robertson, *Rev. Mod. Phys.* 5 (1933), 62.
‡ Loc. cit. (1946).

(*b*) *Milne's substratum.* It has been shown by Robertson† and Walker‡ that there exists a V_4 for the system of observers of Milne's system, having the metric (1) with $R(\tau) = c\tau$, $k = -1$. The density of fundamental particles is proportional to τ^{-3} and the path of a free test particle is given by equations (9) with $\Gamma = -\tau^{-1}$. The motion of the system of fundamental particles is of a hydrodynamic character for which the pressure of matter is taken to be zero. These properties of Milne's substratum are also the properties of our model M with $R(\tau) = c\tau$, $k = -1$, provided that we identify the path of a free test particle in Milne's system with a free path of the model M. From the theory of regraduation we then have the result that:

The kinematic models of class $M(R, \tau, -1)$ are equivalent to Milne's substratum.

From the results of section (*a*) above we can find Lemaître models which are equivalent to Milne's substratum. For this purpose we shall suppose that the Lemaître models are defined with respect to the metric (2) and we consider the case for which the cosmical constant $\lambda = 0$. From the equation similar to (28) for the metric (2), we find on setting $k = -1$, $\lambda = 0$ that

$$\frac{d\bar{R}}{d\bar{\tau}} = \sqrt{\left(c^2 + \frac{\kappa A}{3\bar{R}}\right)}, \tag{29}$$

where the positive sign is chosen to give the solution for an expanding universe. From equations (22) and (23) we have, when $R(\tau) = c\tau$,

$$\frac{d\bar{\tau}}{\bar{R}} = \frac{d\tau}{c\tau}. \tag{30}$$

Then from (29) and (30) we easily find

$$\bar{R}(\bar{\tau}) = \frac{\kappa A}{12\tau_0 c^2} \frac{(\tau - \tau_0)^2}{\tau}, \tag{31}$$

$$\bar{\tau} - \bar{\tau}_0 = \frac{\kappa A}{12\tau_0 c^3} (\tau - 2\tau_0 \log \tau - \tau_0^2 \tau^{-1}), \tag{32}$$

where τ_0, $\bar{\tau}_0$ are constants of integration.

In the limiting case $\tau_0 \to 0$, $A \to 0$, $A\tau_0^{-1} \to 1$, $\bar{\tau}_0 = 0$, we find from (31) and (32) that $\bar{R}(\bar{\tau}) = c\bar{\tau}$, and the metric of the Lemaître model is the same as that of Milne's substratum. Milne's system is therefore included amongst the Lemaître models when the density tends to zero and the cosmical constant $\lambda = 0$. This result was first obtained by

† Loc. cit. (1935).

‡ *Proc. London Math. Soc.* **42** (1936), 90.

Kermack and McCrea.† Also from (31) and (32), we have the more general result, that models equivalent to Milne's system are included amongst the Lemaître models for any value of A.

(c) *Dirac's model.* We next consider the kinematic models M and $\bar{M}$ defined with respect to the metrics (1) and (2) and having the properties

$$R(\tau) = c\tau^{\frac{1}{2}}, \quad k = 0, \qquad \rho = A/c^3\tau, \quad p = 0, \tag{33}$$

$$\bar{R}(\bar{\tau}) = c(2\bar{\tau})^{\frac{1}{2}}, \quad k = 0, \qquad \bar{\rho} = A/4c^3\bar{\tau}^2, \quad \bar{p} = 0, \tag{34}$$

respectively. From (22) and (23) we find that M can be transformed into $\bar{M}$ by taking

$$\sigma(\tau) = \log\tau, \qquad \bar{\tau} = \tfrac{1}{2}\tau^2. \tag{35}$$

The equations (33) also give the properties of the model determined by Dirac‡ from considerations based on the occurrence in nature of certain large dimensionless numbers of the order of 10^{39}. Dirac finds for this model that the ratio γ of the gravitational force to the electric force between an electron and a proton varies inversely as τ. When

$$\kappa = 16c^3/3A,$$

the equations (34) give the properties of a Lemaître model, which was also found by Dirac by changing the units of time and distance measurements so that the ratio γ has a value $\bar{\gamma}$ which is independent of the epoch. This transformation used by Dirac is similar to the transformation of local coordinates given by (19) and (35). Dirac's model can therefore be transformed into a Lemaître model by regraduating the local coordinates, and also more generally:

The kinematic models $M(R, \tau, 0)$ are equivalent to Dirac's model.

7. Summary

A 'kinematic system' is defined for the general space-time of relativistic cosmology. It consists of a system of fundamental observers together with a statistical system of material particles. In the limiting case when the pressure of matter is negligibly small the kinematic system determines a class of kinematic models of the universe. Included in this class of models are the Lemaître models of zero pressure, Milne's substratum, and Dirac's model. Certain kinematic models can be transformed into other models by regraduating the local coordinates. In particular it is shown that Milne's substratum and Dirac's model can be transformed into Lemaître models.

† *M.N.R.A.S.* 93 (1933), 519.

‡ *Proc. Royal Soc.* A, 165 (1938) 199.

CONNECTIVITY THEOREMS FOR GRAPHS

By G. A. DIRAC (*London*)

[Received 1 June 1951]

By 'graph' we mean in this paper a set $\mathscr{N}$ whose members are called the *nodes* together with a set $\mathscr{E}$ of unordered pairs of unequal members of $\mathscr{N}$ called the *edges*. Two disjoint sets of nodes of a graph are said to be *separated* if there is no way leading from a node of one of the sets to a node of the other. The following very general theorem is contained in a paper by Menger (**1**).

Let S_1 and S_2 be two non-empty disjoint sets of nodes of a graph which are not separated. If S_1 and S_2 cannot be separated either by deleting from the graph any $d-1$ nodes which do not belong to S_1 or to S_2, or by deleting from the graph any $d-1$ edges, then the graph contains at least d ways having the following properties:

(i) *each way connects a node of S_1 and a node of S_2 and no way has an intermediate node in common with S_1 or with S_2;*

(ii) *no two ways have an intermediate node in common.*

A short proof has been given by Hajós (**2**) and an account will be found in König (**3**). This paper contains three extensions of Menger's theorem in the case $d = 2$; it will also be shown that it is not possible to make analogous extensions for $d \geqslant 3$.

A way having the property (i) will be called an *$S_1 S_2$-way*. If a node or an edge is such that all $S_1 S_2$-ways contain it, then it will be said to *separate* S_1 and S_2. In this terminology Menger's theorem for the case $d = 2$ states that:

If S_1 and S_2 are two non-empty disjoint sets of nodes of a graph Γ which are not separated by a node or an edge, then Γ contains two $S_1 S_2$-ways which have no intermediate node in common (**4**). An extension of this is contained in the following:

THEOREM 1. *If x and y are two nodes of a graph which are not separated by a node or an edge, and if W is any given way connecting x and y, then the graph contains two xy-ways such that*

(i) *they have no intermediate node in common;*

(ii) *as we go along each of them from x to y, the nodes of W occur in the same order as they do on W as we go from x to y.*

It is convenient to introduce the following notation: If $\mathscr{V}$ is a way which contains the nodes g and h, the portion of $\mathscr{V}$ connecting g and h

will be denoted by $\mathscr{V}(g,h)$. If $g = h$, then $\mathscr{V}(g,h)$ is simply the node g. If the ways $\mathscr{V}'$ and $\mathscr{V}''$ have one end node in common but are otherwise disjoint, then the way which consists of $\mathscr{V}'$ and $\mathscr{V}''$ together will be denoted by $\mathscr{V}'+\mathscr{V}''$.

To prove the theorem I use induction on the length of the given way W. The theorem is clearly true for a pair of nodes for which the given way consists of one edge. We assume that the theorem is true for any pair of nodes if the length of the given way between them does not exceed n and we deduce that it is true for any pair of nodes if the length of the given way between them is $n+1$.

Let a_0 and a_{n+1} be a pair of nodes joined by a way W whose nodes in order are $a_0, a_1, a_2, \ldots, a_{n+1}$ $(n \geqslant 1)$. By hypothesis a_0 and a_n are connected by two ways W_1 and W_2 having the properties (i) they have no node other than a_0 and a_n in common, (ii) as we go along each of them from a_0 to a_n the nodes of $W(a_0, a_n)$ occur in the same order as they do if we go along W from a_0 to a_n. (One of them may, of course, be $W(a_0, a_n)$ itself.) We may suppose that neither W_1 nor W_2 goes through a_{n+1}; otherwise there is nothing to prove. Now a_n does not separate a_0 and a_{n+1}, so there is a way connecting a_0 and a_{n+1} which does not go through a_n. This way has a part which connects one of the nodes $a_0, a_1, \ldots, a_{n-1}$ (a_m say) with a_{n+1}, and has no other nodes from among $a_0, a_1, \ldots, a_{n-1}$. We denote this part by $\mathscr{W}$; it is possible that $\mathscr{W}$ coincides with a part of W_1 or W_2. Let a_p $(0 \leqslant p \leqslant m)$ be the node with greatest suffix not exceeding m among $a_0, a_1, \ldots, a_{n-1}$ which belongs to W_1 or to W_2, and suppose (without loss of generality) that it belongs to W_1; then W_2 has no node in common with $W(a_p, a_m)$. The nodes a_0 and a_{n+1} are therefore connected by the two ways

$$W_1(a_0, a_p)+W(a_p, a_m)+\mathscr{W} \quad \text{and} \quad W_2+W(a_n, a_{n+1}),$$

and these ways satisfy the requirements of the theorem, which is therefore proved.

From Theorem 1 we can immediately deduce the

COROLLARY. *If x is joined to a node z of W, then the graph contains two ways connecting x and y such that they have the properties* (i) *and* (ii) *and one of them goes through z.*

It is easy to see that the theorem remains true if instead of the two nodes x and y we have two disjoint sets of nodes S_1 and S_2 and W is any $S_1 S_2$-way. The analogous extension of Theorem 1 for $d \geqslant 3$ would be the following: If x and y are two nodes of a graph Γ which are connected by a way W, and which cannot be separated either by deleting from the

graph any $d-1$ ($d \geqslant 3$) nodes, or by deleting from the graph any $d-1$ edges, then the graph contains d xy-ways such that (i) no two have an intermediate node in common, (ii) as we go along each of them from x to y, the nodes of W occur in the same order as they do on W when we go from x to y. However, this is not true, not even for $d = 3$. An example is shown in Fig. 1, the edges of W being marked thick.

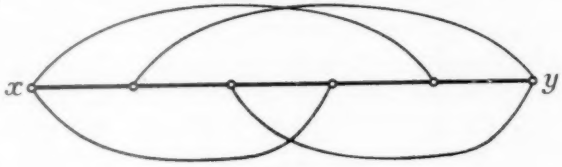

FIG. 1.

A node which separates two nodes will be called a *cut node*. The second extension of Menger's theorem is then as follows:

THEOREM 2. *If S_1 and S_2 are two non-empty disjoint sets of nodes of a graph Γ which are not separated by a node or an edge, and if S_1 and S_2 each contain at least two nodes and no cut node, then the graph contains two S_1S_2-ways which have neither an intermediate nor an end node in common.*

By Theorem 1, Γ contains two S_1S_2-ways, W_1 and W_2 having no intermediate node in common. If they have no end node in common, there is nothing to prove; so we suppose that they have one or both end nodes in common.

Case 1. *W_1 and W_2 have both end nodes in common*

Let the two end nodes be a_1 and a_2, a_1 being in S_1 and a_2 in S_2. Since a_1 is not a cut node, there is a way $\mathscr{W}$ connecting a_2 with a node b_1 ($\neq a_1$) of S_1 which has no intermediate node in common with S_1. Going along $\mathscr{W}$ from a_2 to b_1, let c be the last intersection of $\mathscr{W}$ with W_1 and W_2, and suppose that c belongs to W_1. Then Γ contains W_2 and the way

$$W_1(a_2, c)+\mathscr{W}(c, b_1);$$

these ways have only a_2 in common. If $c \neq a_2$, there is the possibility that $\mathscr{W}$ passes through another node of S_2 besides a_2; in this case there is a node a_2' ($\neq a_2$) belonging to S_2 such that W_2 and $\mathscr{W}(a_2', b_1)$ satisfy the requirements of the theorem. In all other cases the two ways W_2 and $W_1(a_2, c)+\mathscr{W}(c, b_1)$ are S_1S_2-ways and have only a_2 in common; this situation is equivalent to:

Case 2. *W_1 and W_2 have only the end node a_2 in common*

Let the end nodes of W_1 and W_2 in S_1 be a_1 and b_1 respectively. Since

a_2 is not a cut node, Γ contains an $S_1 S_2$-way $\mathscr{V}$ connecting a node c_1 of S_1 with a node c_2 of S_2, where $c_2 \neq a_2$ but possibly $c_1 = a_1$ or $c_1 = b_1$. If $\mathscr{V}$ has no node in common with one of the ways W_1 or W_2, then $\mathscr{V}$ and the other one together satisfy the requirements of the theorem. If $\mathscr{V}$ has a node in common with each of the ways W_1 and W_2, suppose d to be the last intersection of $\mathscr{V}$ with W_1 and W_2 as we go along $\mathscr{V}$ from c_1 to c_2, and suppose that d belongs to W_1. Then the two ways W_2 and $W_1(a_1, d)+\mathscr{V}(d, c_2)$ satisfy the requirements of the theorem. This completes the proof.

This result cannot be extended to the cases $d \geqslant 3$. The graph illustrated is such that any two nodes can be deleted and it remains connected, and S_1 and S_2 each contain not less than three nodes, but the graph does not contain three ways connecting a node of S_1 with a node of S_2 no two of which have a node in common.

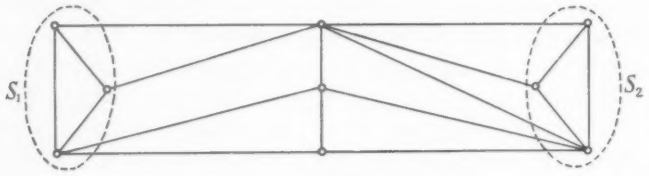

FIG. 2.

Finally it is easy to see that Theorems 1 and 2 give the following

THEOREM 3. *If S_1 and S_2 are two disjoint non-empty sets of nodes of a graph which are not separated by a node or an edge and each contain at least two nodes and no cut node, and if W is any given $S_1 S_2$-way, then the graph contains two completely disjoint $S_1 S_2$-ways such that as we go along each of them from the end in S_1 to the end in S_2, the nodes of W, if they occur at all, occur in the same order as they do on W as we go from the end in S_1 to the end in S_2.*

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SELF-ADJOINT DIFFERENTIAL SYSTEMS

By K. S. MILLER (*New York*)

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1. Introduction

D. JACKSON (**1**) has given a criterion that an ordinary linear differential system be self-adjoint. These conditions were later written in a more explicit, though less symmetric, form by V. V. Latshaw (**2**). It is my purpose in this paper to obtain a new criterion which does not make use of the Lagrange bilinear concomitant, but rather is based on a consideration of the Green's function of the system. While in some respects it is less explicit than the results of Jackson, it is more convenient to apply in certain circumstances.

I shall suppose that we have the linear differential system

$$\left.\begin{aligned} Lu &= p_0(x)u^{(n)}+p_1(x)u^{(n-1)}+\ldots+p_n(x)u \\ U_\alpha(u) &= V_\alpha(u)+Z_\alpha(u) \quad (\alpha = 1,2,\ldots,n) \end{aligned}\right\} \tag{1}$$

with $$V_\alpha(u) = \sum_{i=0}^{n-1} A_{\alpha i}\, u^{(i)}(a), \qquad Z_\alpha(u) = \sum_{i=0}^{n-1} B_{\alpha i}\, u^{(i)}(b),$$

where the coefficients $p_i(x)$ are of class C^{n-i} $(i = 0, 1, 2,\ldots, n)$ in the closed finite interval (a, b) and $p_0(x) > 0$ in (a, b). Furthermore I shall suppose that the completely homogeneous system

$$Lu = 0, \qquad U_\alpha(u) = 0 \quad (\alpha = 1, 2,\ldots, n) \tag{2}$$

is incompatible.

The condition that I shall obtain for self-adjointness of the differential system (1) will be that a certain matrix be symmetric or skew-symmetric. In order to facilitate the discussion, I shall introduce the following notation: $F = ||f(i,j)||$ is to be an $n\times n$ matrix with the element $f(i,j)$ in the ith row and jth column; $F' = ||f(j,i)||$ will denote the transpose of F; $|F|$ will be the determinant of F. If

$$\{\phi_1(x), \phi_2(x),\ldots, \phi_n(x)\}$$

are n linearly independent solutions of $Lu = 0$, I shall write

$$U_\alpha(\phi_\beta) = V_\alpha(\phi_\beta)+Z_\alpha(\phi_\beta)$$

as $U_{\alpha\beta} = V_{\alpha\beta} + Z_{\alpha\beta}$ $(\alpha, \beta = 1, 2, \ldots, n)$ and shall define $\Delta_\alpha(\phi_\beta)$ as

$$V_\alpha(\phi_\beta) - Z_\alpha(\phi_\beta),$$

i.e.
$$\Delta_{\alpha\beta} = V_{\alpha\beta} - Z_{\alpha\beta} \quad (\alpha, \beta = 1, 2, \ldots, n). \tag{3}$$

I now define the following matrices:

$$D = ||U_{\alpha\beta}||, \qquad \Delta = ||\Delta_{\alpha\beta}||, \qquad V = ||V_{\alpha\beta}||, \qquad Z = ||Z_{\alpha\beta}||. \tag{4}$$

2. The Green's function

The Green's function $G(x, \zeta)$ for the system (2) can now be written [(3) 189] as

$$G(x, \zeta) = \frac{1}{|D|} \begin{vmatrix} g(x, \zeta) & \phi_1(x) & \cdots & \phi_n(x) \\ U_1(g) & U_{11} & \cdots & U_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ U_n(g) & U_{n1} & \cdots & U_{nn} \end{vmatrix}, \tag{5}$$

where, by (3) 75,

$$g(x, \zeta) = \frac{(-1)^n \sigma}{p_0(\zeta) W(\zeta)} \begin{vmatrix} \phi_1(x) & \phi_2(x) & \cdots & \phi_n(x) \\ \phi_1(\zeta) & \phi_2(\zeta) & \cdots & \phi_n(\zeta) \\ \phi_1'(\zeta) & \phi_2'(\zeta) & \cdots & \phi_n'(\zeta) \\ \cdots & \cdots & \cdots & \cdots \\ \phi_1^{(n-2)}(\zeta) & \phi_2^{(n-2)}(\zeta) & \cdots & \phi_n^{(n-2)}(\zeta) \end{vmatrix}. \tag{6}$$

Here $W(\zeta)$ is the Wronskian of $\{\phi_1(\zeta), \phi_2(\zeta), \ldots, \phi_n(\zeta)\}$ and $\sigma = +\frac{1}{2}$ if $x < \zeta$ and $-\frac{1}{2}$ if $x > \zeta$.

A necessary and sufficient condition [(5) 256] that the system (1) be self-adjoint is that $G(x, \zeta) = (-1)^n G(\zeta, x)$. I shall proceed to apply this criterion. First I prove a lemma.

LEMMA. Hypothesis: *Let the differential operator L be formally self-adjoint: that is, $L = (-1)^n L^+$ where L^+ is the Lagrange adjoint of L. Let $\{\phi_1(x), \phi_2(x), \ldots, \phi_n(x)\}$ be n linearly independent solutions of $Lu = 0$ with Wronskian $W(x)$, and let $\{\phi_1^*(x), \phi_2^*(x), \ldots, \phi_n^*(x)\}$ be the adjoint system of solutions of $L^+u = 0$. Let $C = ||c_{ij}||$ be the matrix which takes the $\phi_i^*(x)$ into the $\phi_i(x)$: that is $||\phi_i^*(x)|| = ||c_{ij}|| \cdot ||\phi_j(x)||$, where $||\phi_i^*||$ and $||\phi_j||$ are column vectors.*

Conclusion: *$C = (-1)^{n+1} C'$, i.e. C is symmetric (skew-symmetric) if L is a differential operator of odd (even) order.*

Proof. If we expand $g(x, \zeta)$ by minors of the first row, we obtain

$$g(x, \zeta) = -\sigma \sum_{i=1}^{n} \phi_i(x) w_i(\zeta), \tag{7}$$

where
$$w_i(\zeta) = \frac{1}{p_0(\zeta)} \frac{\partial}{\partial \phi_i^{(n-1)}} \log W(\zeta). \tag{8}$$

But $\phi_i^*(x) = w_i(x)$ by (4) 102. Hence (7) can be written

$$g(x,\zeta) = -\sigma \sum_{i=1}^{n} \phi_i(x)\phi_i^*(\zeta). \tag{9}$$

Since $L = (-1)^n L^+$, the $\phi_i^*(x)$ are solutions of $Lu = 0$. Hence the ϕ_i^* are linear combinations of the ϕ_j and $\phi_i^*(x) = \sum_{j=1}^{n} c_{ij}\phi_j(x)$ $(i = 1, 2, \ldots, n)$ with non-singular matrix $C = ||c_{ij}||$. Again, since $L = (-1)^n L^+$, we conclude that $g(x,\zeta) = (-1)^n g(\zeta, x)$ since $g(x,\zeta)$ depends only on L and not on the boundary conditions $U_\alpha(u)$. Hence it follows from (9) that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\phi_i(x)\phi_j(\zeta) = (-1)^{n+1}\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ji}\phi_j(\zeta)\phi_i(x),$$

i.e. $c_{ij} = (-1)^{n+1}c_{ji}$.

3. The main theorem

THEOREM. *A necessary and sufficient condition that the differential system* (1) *be self-adjoint is that* $\Delta CD'$ *be symmetric* (*skew-symmetric*) *if* L *is a formally self-adjoint differential operator of even* (*odd*) *order.*

Proof. In (5),

$$U_\alpha(g) = \sum_{j=1}^{n} a_{\alpha j}\phi_j(\zeta) \quad (\alpha = 1, 2, \ldots, n), \tag{10}$$

where
$$a_{\alpha j} = -\tfrac{1}{2}\sum_{i=1}^{n} \Delta_{\alpha i} c_{ij} \quad (\alpha, j = 1, 2, \ldots, n). \tag{11}$$

I shall denote the function of ζ in (10) by $\psi_\alpha(\zeta)$.

Since $G(x,\zeta) = (-1)^n G(\zeta, x)$ is necessary and sufficient that (1) be self-adjoint, we must have

$$\begin{vmatrix} g(x,\zeta) & \phi_1(x) & \cdots & \phi_n(x) \\ \psi_1(\zeta) & U_{11} & \cdots & U_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ \psi_n(\zeta) & U_{n1} & \cdots & U_{nn} \end{vmatrix}$$

$$= (-1)^n \begin{vmatrix} g(\zeta,x) & \phi_1(\zeta) & \cdots & \phi_n. \\ \psi_1(x) & U_{11} & \cdots & U_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ \psi_n(x) & U_{n1} & \cdots & U_{nn} \end{vmatrix}.$$

Noting that $g(x,\zeta) = (-1)^n g(\zeta, x)$ since $L = (-1)^n L^+$ and letting $u_{\alpha\beta}$ be the cofactor (signed minor) of $U_{\alpha\beta}$ in the matrix $D = ||U_{\alpha\beta}||$, we

find, on expanding the above determinants by minors of the first row, that

$$\sum_{j=1}^{n}\sum_{\beta=1}^{n}\Big[\sum_{\alpha=1}^{n}a_{\alpha j}\,u_{\alpha\beta}\Big]\phi_\beta(x)\phi_j(\zeta)=\sum_{j=1}^{n}\sum_{\beta=1}^{n}\Big[(-1)^n\sum_{\alpha=1}^{n}a_{\alpha\beta}\,u_{\alpha j}\Big]\phi_\beta(x)\phi_j(\zeta),$$

i.e. $\sum_{\alpha=1}^{n}a_{\alpha j}\,u_{\alpha\beta}=\sum_{\alpha=1}^{n}(-1)^n a_{\alpha\beta}\,u_{\alpha j}$ $(\beta,\ j=1,2,\ldots,n)$. In matrix notation, the above equations become

$$A'U=(-1)^nU'A, \tag{12}$$

where $U=||u_{\alpha\beta}||$ and $A=||a_{\alpha\beta}||$.

From the definition of U we see that $DU'=UD'=|D|I$, where $I=||\delta_{ij}||$ is the unit matrix. Also from (11) we deduce $A=-\frac{1}{2}\Delta C$. Utilizing these results in (12) yields

$$\Delta CD'=(-1)^nDC'\Delta'=(-1)^n(\Delta CD')', \tag{13}$$

from which the theorem follows.

Example: $n=2$. We know [(5) 216] that for a second-order linear differential system to be self-adjoint it is necessary and sufficient that $|V|=|Z|$. Written in expanded form this becomes

$$U_{11}\Delta_{22}+\Delta_{11}U_{22}=U_{12}\Delta_{21}+U_{21}\Delta_{12}.$$

The theorem says that

$$\Delta CD'=\begin{Vmatrix}\Delta_{11}&\Delta_{12}\\ \Delta_{21}&\Delta_{22}\end{Vmatrix}.\begin{Vmatrix}c_{11}&c_{12}\\ c_{21}&c_{22}\end{Vmatrix}.\begin{Vmatrix}U_{11}&U_{21}\\ U_{12}&U_{22}\end{Vmatrix}$$

should be symmetric. With the aid of (8) we find that $C=a\begin{Vmatrix}0&1\\ -1&0\end{Vmatrix}$, where a is a non-zero constant, and readily verify that $\Delta CD'=(\Delta CD')'$ implies $U_{11}\Delta_{22}+\Delta_{11}U_{22}=U_{12}\Delta_{21}+U_{21}\Delta_{12}$. [In general C can be reduced to the form $C=a||(-1)^{i+1}\delta_{i,n+1-j}||$ $(i,j=1,2,\ldots,n)$, where a is a non-zero constant, by a suitable choice of the fundamental system $\{\phi_1(x),\phi_2(x),\ldots,\phi_n(x)\}$. See (4) 114.]

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TWO IDENTITIES DUE TO RAMANUJAN

By H. F. SANDHAM (*Dublin*)

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1. WRITING, as usual,

$$q_0 = (1-q^2)(1-q^4)(1-q^6)\ldots, \qquad q_1 = (1+q^2)(1+q^4)(1+q^6)\ldots,$$
$$q_2 = (1+q)(1+q^3)(1+q^5)\ldots, \qquad q_3 = (1-q)(1-q^3)(1-q^5)\ldots,$$

and writing for convenience

$$Q_0, Q_1, Q_2, Q_3$$

for the same expressions when q is replaced by q^3, I prove the identities (1)†

$$q^{1/3}q_0^3/q_2^2 = 1q^{1^2/3} - 2q^{2^2/3} + 4q^{4^2/3} - 5q^{5^2/3} + \\ + 7q^{7^2/3} - 8q^{8^2/3} + 10q^{10^2/3} - 11q^{11^2/3} + \ldots,$$

$$q^{1/24}q_0^3 q_3^5 = 1q^{1^2/24} - 5q^{5^2/24} + 7q^{7^2/24} - 11q^{11^2/24} + \\ + 13q^{13^2/24} - 17q^{17^2/24} + 19q^{19^2/24} - 23q^{23^2/24} + \ldots.$$

From these I deduce two series of partitions

$$\sum_0^\infty \frac{p(n)}{2\cosh \frac{1}{3}\pi\sqrt{(24n-1)}+1} = \frac{\log(\sqrt{2}+1)}{\sqrt{3}},$$

$$\sum_1^\infty \frac{p(n)}{2\cosh \frac{1}{3}\pi\sqrt{(24n-1)}-1} = \frac{\sqrt{3}}{2\pi} - \frac{1}{6} - \frac{\log 2}{\sqrt{3}}.$$

2. The identities are proved as follows. Writing

$$f(x) = q_0(1+qx)\left(1+\frac{q}{x}\right)(1+q^3x)\left(1+\frac{q^3}{x}\right)\ldots = \sum_{-\infty}^{+\infty} q^{n^2}x^n,$$

we see at once that we can write

$$f^3(wx) - f^3(w^2x) = A(x-1/x) + B(x^2-1/x^2) + 0 + D(x^4-1/x^4) + \ldots,$$

where

$$w = \exp \tfrac{2}{3}\pi i.$$

From the functional relation

$$f(q^2x) = f(x)/qx$$

we see that

$$f^3(q^2wx) - f^3(q^2w^2x) = \{f^3(wx) - f^3(w^2x)\}/q^3x^3,$$

† The writer discovered these identities while studying different ways of writing the modular equation of order 3, but later found they had been stated by Ramanujan.

whence B, D, E, G,... are determined in terms of A, and we find

$$f^3(wx)-f^3(w^2x)$$
$$= A\sum_0^\infty q^{3n^2+2n}(x^{3n+1}-1/x^{3n+1})-q^{3n^2+4n+1}(x^{3n+2}-1/x^{3n+2}).$$

Letting $x=-q$ we determine A, and thus derive

$$f^3(wx)-f^3(w^2x) = 3i\sqrt{3}q\frac{Q_0^3}{q_0}\sum_0^\infty q^{3n^2+2n}(x^{3n+1}-1/x^{3n+1})- \\ -q^{3n^2+4n+1}(x^{3n+2}-1/x^{3n+2}).$$

Divide across by $x-1/x$ and let $x\to 1$; on the right-hand side we have

$$3i\sqrt{3}q\frac{Q_0^3}{q_0}\sum_0^\infty q^{3n^2+2n}(3n+1)-q^{3n^2+4n+1}(3n+2),$$

which it is convenient to write

$$6iq^{2/3}\frac{Q_0^3}{q_0}\sum_1^\infty q^{n^2/3}n\sin\tfrac{2}{3}n\pi.$$

On the left-hand side there is

$$\lim_{x\to 1}\frac{f^3(wx)-f^3(w^2x)}{x-1/x} = \lim_{x\to 1}\frac{f(wx)-f(w^2x)}{x-1/x}\,3q_0^2\frac{Q_2^2}{q_2^2}$$
$$= 6i\frac{q_0^2}{q_2^2}\frac{Q_2^2}{q_2^2}\sum_1^\infty q^{n^2}n\sin\tfrac{2}{3}n\pi.$$

Hence
$$\frac{\sum_1^\infty q^{n^2/3}n\sin\frac{2}{3}n\pi}{q^{1/3}q_0^3/q_2^2} = \frac{\sum_1^\infty q^{n^2}n\sin\frac{2}{3}n\pi}{qQ_0^3/Q_2^2}.$$

Replacing q by q^3, q^9, q^{27},... now yields the first identity.

Since
$$q_0q_2^2 = \sum_{-\infty}^{+\infty} q^{n^2},$$

therefore
$$q_0q_2^2+q_0q_3^2 = 2q_0(q^4)q_2^2(q^4).$$

Hence, changing the sign of q in the first identity and adding, we deduce the second identity.

3. To establish the first series of partitions, we integrate termwise the first identity, and find

$$\int_0^1 q^{\alpha+1/3}\frac{q_0^3}{q_2^2}\frac{dq}{q} = 2\sqrt{3}\sum_1^\infty\frac{n\sin\frac{2}{3}n\pi}{n^2+3\alpha^2} = \frac{\pi\sqrt{3}}{2\cosh(2\pi\alpha/\sqrt{3})+1} \quad (\alpha>0).$$

Now $$\sum_0^\infty p(n)q^{2n} = 1/q_0.$$

Hence $$\int_0^1 q^{1/4}\frac{q_0^2}{q_2^2}\frac{dq}{q} = \pi\sqrt{3}\sum_0^\infty \frac{p(n)}{2\cosh\frac{1}{3}\pi\sqrt{(24n-1)}+1}.$$

A formula of Jacobi's (**2**)

$$q^{1/2}q_0^2q_1^4 = \sum_0^\infty \frac{q^{n+1/2}}{1+q^{2n+1}},$$

which tells us the number of ways a number is the sum of two triangular numbers, may be written

$$q^{1/4}\frac{q_0^2}{q_2^2} = \sum_0^\infty (-1)^n\left(\frac{q^{n+1/4}}{1+q^{4n+1}} - \frac{q^{3n+9/4}}{1+q^{4n+3}}\right).$$

Hence

$$\pi\sqrt{3}\sum_0^\infty \frac{p(n)}{2\cosh\frac{1}{3}\pi\sqrt{(24n-1)}+1} = \sum_0^\infty (-1)^n\int_0^1\left(\frac{q^{n+1/4}}{1+q^{4n+1}} - \frac{q^{3n+9/4}}{1+q^{4n+3}}\right)\frac{dq}{q}$$

$$= \sum_0^\infty \frac{(-1)^n}{n+\frac{1}{4}}\int_0^1 \frac{q}{1+q^4}\frac{dq}{q} - \sum_0^\infty \frac{(-1)^n}{n+\frac{3}{4}}\int_0^1 \frac{q^3}{1+q^4}\frac{dq}{q}$$

$$= \pi\log(\sqrt{2}+1).$$

Similarly we establish the second series:

$$\int_0^1 q^{\alpha^2+1/24}q_0^3q_3^5\frac{dq}{q} = \frac{4\sqrt{3}}{2\cosh 2\pi\alpha\sqrt{\frac{2}{3}}-1},$$

$$4\pi\sqrt{3}\sum_0^\infty \frac{p(n)}{2\cosh\frac{1}{3}\pi\sqrt{(24n+24\alpha^2-1)}-1} = \int_0^1 q^{\alpha^2}q_0^2q_3^4\frac{dq}{q}.$$

This time, applying Jacobi's formula (**2**)

$$q_0^2q_2^4 = 1+4\sum_0^\infty (-1)^n\frac{q^{2n+1}}{1-q^{2n+1}}$$

telling us the number of ways a number is the sum of two squares, we write this

$$\frac{4\pi\sqrt{3}}{2\cosh\frac{1}{3}\pi\sqrt{(24\alpha^2-1)}-1} + 4\pi\sqrt{3}\sum_1^\infty \frac{p(n)}{2\cosh\frac{1}{3}\pi\sqrt{(24n+24\alpha^2-1)}-1}$$

$$= \frac{1}{\alpha^2} - 4\sum_0^\infty (-1)^n\int_0^1 q^{\alpha^2}\frac{q^{2n+1}}{1+q^{2n+1}}\frac{dq}{q}.$$

Letting $\alpha \to 0$, this becomes

$$4\pi\sqrt{3}\sum_{1}^{\infty}\frac{p(n)}{2\cosh\frac{1}{3}\pi\sqrt{(24n-1)}-1}=6-\frac{2\pi}{\sqrt{3}}-4\sum_{0}^{\infty}\frac{(-1)^n}{2n+1}\int_{0}^{1}\frac{q}{1+q}\frac{dq}{q},$$

whence the second series is established.

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A THEOREM ON SEQUENCES OF CONVEX SETS

By R. RADO (*London*)

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1. This note is concerned with a joint application of two remarkable general theorems due to F. P. Ramsey [(**1**), Theorem A] and E. Helly (**2**) respectively. The theorems are:

RAMSEY'S THEOREM. *Suppose that all sets of r positive integers are divided into a finite number of classes. Then there exists an infinite set M of positive integers such that all sets of r numbers of M belong to the same class.*

HELLY'S THEOREM. *Let $T_1, T_2, \ldots, T_N$ be convex sets of points in euclidean n-dimensional space ($N \geqslant n+2$). Suppose that any $n+1$ of the sets T_ν have a point in common. Then all N sets T_ν have a point in common.*

The result (Theorem 1) is a proposition which states that any arbitrary sequence of convex sets in a euclidean n-space contains a sub-sequence which behaves in an extremely uniform way as far as emptiness or non-emptiness of intersections of members of the sub-sequence is concerned. Either there exists a number k ($0 \leqslant k \leqslant n$) such that all such intersections of k members are non-empty and all intersections of $k+1$ members are empty, or else all intersections of any finite number of such members are non-empty.

Theorem 2 is obtained from Theorem 1 by a familiar type of finitization in which infinite sequences are replaced by finite systems. Finally, Theorem 3 states the existence of sequences of closed, convex, and uniformly bounded sets of a very simple kind which exhibit the intersection behaviour described in Theorem 1, corresponding to any given number k in the range $0 \leqslant k \leqslant n$.

2. THEOREM 1. *Let $S_1, S_2, S_3, \ldots$ be a sequence of convex sets of points in n-dimensional euclidean space. Then there is an increasing sequence of positive integers $\alpha_1, \alpha_2, \ldots$ such that the sets $T_\nu = S_{\alpha_\nu}$ have the following property. Either* (i) *there is an integer k in the range $0 \leqslant k \leqslant n$ such that*†

$$T_{\rho_1} T_{\rho_2} \ldots T_{\rho_k} \neq 0, \qquad T_{\rho_1} \ldots T_{\rho_k} T_{\rho_{k+1}} = 0$$

† If $k = 0$ then the assertion means that all T_ν are empty.

if $1 \leqq \rho_1 < \rho_2 < \ldots < \rho_{k+1}$, *or* (ii) $T_1 T_2 \ldots T_N \neq 0$ *for all* $N > 0$. *If, in addition, each set* S_ν *is closed, then, in case* (ii), *either none of the sets* $T_1 T_2 \ldots T_N$ *is bounded* $(N > 0)$, *or* $T_1 T_2 T_3 \ldots \neq 0$.

THEOREM 2. *Given positive integers* n *and* N, *there exists a positive integer* $f = f(n, N)$ *which has the following property. Given any* f *convex sets of points* $S_1, S_2, \ldots, S_f$ *in* n*-dimensional euclidean space, there are* N *integers* α_ν *satisfying* $1 \leqq \alpha_1 < \alpha_2 < \ldots < \alpha_N \leqq f$, *such that, if* $T_\nu = S_{\alpha_\nu}$, *either* (i) *there is an integer* k *in the range* $0 \leqq k \leqq n$ *satisfying*

$$T_{\rho_1} T_{\rho_2} \ldots T_{\rho_k} \neq 0 \quad \text{for} \quad 1 \leqq \rho_1 < \rho_2 < \ldots < \rho_k \leqq N, \dagger$$

$$T_{\rho_1} \ldots T_{\rho_k} T_{\rho_{k+1}} = 0 \quad \text{for} \quad 1 \leqq \rho_1 < \rho_2 < \ldots < \rho_{k+1} \leqq N,$$

or (ii) $T_1 T_2 \ldots T_N \neq 0$.

THEOREM 3. *Given integers* k *and* n $(1 \leqq k \leqq n)$, *there exists a sequence of uniformly bounded, closed, convex polyhedra* $S_1, S_2, \ldots$ *in* n*-dimensional euclidean space such that, if* $1 \leqq \rho_1 < \rho_2 < \ldots < \rho_{k+1}$, *the intersection* $S_{\rho_1} S_{\rho_2} \ldots S_{\rho_k}$ *has interior points, and* $S_{\rho_1} \ldots S_{\rho_k} S_{\rho_{k+1}} = 0$. *Each* S_ρ *has at most* $2n+2$ *faces.*

3. *Proof of Theorem* 1. Suppose that the sets S_ν satisfy the hypothesis of Theorem 1. Define a distribution Δ of all sets of $n+1$ positive integers as follows. Two such sets, A and B, belong to the same class of Δ if and only if

$$A = \{a_1, \ldots, a_{n+1}\}, \qquad B = \{b_1, \ldots, b_{n+1}\},$$

$$1 \leqq a_1 < a_2 < \ldots < a_{n+1}, \qquad 1 \leqq b_1 < b_2 < \ldots < b_{n+1},$$

and the following condition holds. The set of all sets

$$\{\kappa, \lambda, \ldots, \mu\} \subset \{1, 2, \ldots, n+1\}$$

such that $S_{a_\kappa} S_{a_\lambda} \ldots S_{a_\mu} = 0$ is identical with the set of all sets

$$\{\kappa, \lambda, \ldots, \mu\} \subset \{1, 2, \ldots, n+1\}$$

such that $S_{b_\kappa} S_{b_\lambda} \ldots S_{b_\mu} = 0$. The number of classes of Δ is finite. By Ramsey's theorem there is an increasing sequence $\alpha_1, \alpha_2, \ldots$ such that all sets $\{\alpha_{\rho_1}, \alpha_{\rho_2}, \ldots, \alpha_{\rho_{n+1}}\}$ $(1 \leqq \rho_1 < \rho_2 < \ldots < \rho_{n+1})$ belong to the same class of Δ. Put $T_\nu = S_{\alpha_\nu}$.

Case 1. Suppose that $T_1 T_2 \ldots T_{n+1} = 0$. Then there is a least number k such that $0 \leqq k \leqq n$, $T_1 T_2 \ldots T_{k+1} = 0$.‡ Now let

$$1 \leqq \rho_1 < \rho_2 < \ldots < \rho_{k+1}.$$

† This line is omitted if $k = 0$.

‡ If $k = 0$, then in what follows all statements about products of k sets have to be omitted.

Then we choose any $n-k$ integers $\rho_{k+2}, \ldots, \rho_{n+1}$ such that

$$\rho_1 < \rho_2 < \ldots < \rho_{n+1}.$$

Then, by definition of the α_ν, the sets $\{\alpha_1, \alpha_2, \ldots, \alpha_{n+1}\}$ and $\{\alpha_{\rho_1}, \alpha_{\rho_2}, \ldots, \alpha_{\rho_{n+1}}\}$ belong to the same class of Δ, and it follows from $T_1 \ldots T_k \neq 0 = T_1 \ldots T_{k+1}$ and the definition of Δ that

$$T_{\rho_1} \ldots T_{\rho_k} \neq 0 = T_{\rho_1} \ldots T_{\rho_{k+1}}.$$

This proves assertion (i) of Theorem 1.

Case 2. Suppose that $T_1 T_2 \ldots T_{n+1} \neq 0$. Let $1 \leqslant \rho_1 < \rho_2 < \ldots < \rho_{n+1}$. Then, by definition of the α_ν, the sets $\{\alpha_1, \ldots, \alpha_{n+1}\}$ and $\{\alpha_{\rho_1}, \ldots, \alpha_{\rho_{n+1}}\}$ belong to the same class of Δ, and hence, by definition of Δ, $T_{\rho_1} \ldots T_{\rho_{n+1}} \neq 0$. Since the ρ_ν are arbitrary, it follows from Helly's theorem that

$$T_1 T_2 \ldots T_N \neq 0 \quad (N > 0),$$

so that assertion (ii) holds.

Now suppose that (ii) is satisfied, that the sets S_ν are closed, and that $T_1 T_2 \ldots T_{N_0}$ is bounded for some N_0. Then $U_N = T_1 T_2 \ldots T_N$, for $N \geqslant N_0$, is a non-increasing sequence of closed, bounded, non-empty sets whose members have, by a well-known theorem, a non-empty intersection. Hence $T_1 T_2 T_3 \ldots \neq 0$, and Theorem 1 is proved.

The proof of Theorem 2 proceeds exactly as that of Theorem 1, the only change being that instead of Ramsey's Theorem A quoted above we use the finitist version of Theorem A which is Theorem B of (**1**).

4. *Proof of Theorem* 3. The construction which follows is similar to one employed in (**3**).

Choose numbers t_ρ such that

$$0 = t_0 < t_1 < t_2 < \ldots, \qquad t_\rho < 1 \quad (\rho > 0),$$

and numbers $\epsilon_\rho > 0$ such that

$$\sum_{\rho=1}^{\infty} (t_{\rho+1} - t_\rho)^{-k} (t_\rho - t_{\rho-1})^{-k} \epsilon_\rho < 1. \tag{1}$$

Let, for $\rho > 0$, S_ρ be the set of all points $x = (x_1, x_2, \ldots, x_n)$ such that

$$|t_\rho^k + x_1 t_\rho^{k-1} + \ldots + x_k| \leqslant \epsilon_\rho, \qquad |x_\nu| \leqslant 2^k \quad (1 \leqslant \nu \leqslant n).$$

Then S_ρ, as intersection of $2+2n$ closed half-spaces, is a closed convex polyhedron which is, moreover, contained in a fixed cube of side 2^{k+1}.

Now consider any $k+1$ indices ρ_κ such that $1 \leqslant \rho_1 < \rho_2 < \ldots < \rho_{k+1}$. Define the point $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ by means of $\bar{x}_\nu = 0$ $(k < \nu \leqslant n)$ and the identity, in t,

$$(t - t_{\rho_1})(t - t_{\rho_2}) \ldots (t - t_{\rho_k}) = t^k + \bar{x}_1 t^{k-1} + \ldots + \bar{x}_k.$$

Then $|\bar{x}_\nu| \leqslant (1+t_{\rho_k})^k < 2^k$ $(1 \leqslant \nu \leqslant n)$, and hence $\bar{x}$ is an interior point of $S_{\rho_1} \dots S_{\rho_k}$.

On the other hand, if $x^* = (x_1^*, \dots, x_n^*) \in S_{\rho_1} \dots S_{\rho_{k+1}}$, then the polynomial $f^*(t) = t^k + x_1^* t^{k-1} + \dots + x_k^*$ satisfies the inequalities

$$|f^*(t_{\rho_\kappa})| \leqslant \epsilon_{\rho_\kappa} \quad (1 \leqslant \kappa \leqslant k+1),$$

as well as the identity, according to Lagrange's interpolation formula,

$$f^*(t) = \sum_{\kappa=1}^{k+1} \frac{(t-t_{\rho_1})\dots(t-t_{\rho_{\kappa-1}})(t-t_{\rho_{\kappa+1}})\dots(t-t_{\rho_{\kappa+1}})}{(t_{\rho_k}-t_{\rho_1})\dots(t_{\rho_\kappa}-t_{\rho_{\kappa-1}})(t_{\rho_\kappa}-t_{\rho_{\kappa+1}})\dots(t_{\rho_\kappa}-t_{\rho_{\kappa+1}})} f^*(t_{\rho_\kappa}),$$

from which we deduce, by comparing the coefficients of t^k, that

$$1 \leqslant \sum_{\kappa=1}^{k+1} (t_{\rho_\kappa}-t_{\rho_\kappa-1})^{-\kappa+1}(t_{\rho_\kappa+1}-t_{\rho_\kappa})^{-k+\kappa-1}\epsilon_{\rho_\kappa} < \sum_{\rho=1}^{\infty} (t_\rho - t_{\rho-1})^{-k}(t_{\rho+1}-t_\rho)^{-k}\epsilon_\rho,$$

i.e. a contradiction against (1). Hence $S_{\rho_1} \dots S_{\rho_{k+1}} = 0$, and the theorem is proved. Each S_ρ is the intersection of a fixed cube and a certain 'strip' between parallel 'planes'.

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NOTE ON A PREVIOUS PAPER ON MATRICES WITH POSITIVE ELEMENTS

By HAZEL PERFECT (*Swansea*)

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In a previous paper (1)† I discussed a method of approximating to the maximal root χ of a positive matrix A; and in § 3 of (1) I made a conjecture concerning the manner in which the sequence of vectors $A^r u$ $(r = 0, 1, 2,...)$, where $u = \{u_1, u_2,..., u_n\}$ is an arbitrary positive vector, approaches the pole $y = \{y_1, y_2,..., y_n\}$ (taken to be a positive vector) of A corresponding to χ. The following numerical example refutes this conjecture.

Example. Let

$$A = \begin{pmatrix} \frac{391}{400} & \frac{1}{400} & \frac{1}{100} & \frac{1}{100} \\ \frac{17}{32} & \frac{7}{32} & \frac{1}{8} & \frac{1}{8} \\ \frac{9}{32} & \frac{15}{32} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{100} & \frac{1}{100} & \frac{4}{100} & \frac{94}{100} \end{pmatrix},$$

and

$$u = \{1, 5, 5, 6\}.$$

Then

$$y = \{1, 1, 1, 1\}, \qquad Au = \{\tfrac{11}{10}, 3, 4, \tfrac{59}{10}\},$$

and

$$\frac{y'Au}{|y| \,.\, |Au|} = \frac{140}{2\times(6102)^{\frac{1}{2}}} < \frac{17}{2\times(87)^{\frac{1}{2}}} = \frac{y'u}{|y| \,.\, |u|},$$

which refutes the conjecture that the sequence

$$\frac{y'A^r u}{|y| \,.\, |A^r u|}$$

is monotonic increasing.

I replace this conjecture by the following theorem.

THEOREM. *The sequence of positive numbers*

$$\frac{\max\limits_{(i)}(e_i' A^r u / e_i' y)}{\min\limits_{(i)}(e_i' A^r u / e_i' y)} \quad (r = 0, 1, 2,...)$$

is a steadily decreasing sequence tending to the limit 1.

† I employ here the notation used in (1).

Proof. The matrix $$C = \frac{1}{\chi} Y^{-1} A Y,$$

where $Y = \operatorname{diag}(y_1, \ldots, y_n)$, is a positive matrix each of whose row sums is equal to 1, i.e. C is a positive stochastic matrix. Now

$$\begin{aligned} e_i' A^r u &= \chi^r e_i' Y C^r Y^{-1} u \\ &= \chi^r (e_i' y) e_i' C^r Y^{-1} u \\ &= \chi^r (e_i' y) e_i' C^r v \end{aligned}$$

where $$v = \{u_1/y_1, \ldots, u_n/y_n\},$$

i.e. $$e_i' A^r u / e_i' y = \chi^r e_i' C^r v.$$

Therefore it remains to prove that the sequence

$$\frac{\max\limits_{(i)} (e_i' C^r v)}{\min\limits_{(i)} (e_i' C^r v)} \quad (r = 0, 1, 2, \ldots)$$

is a steadily decreasing sequence tending to the limit 1.

That this sequence is steadily decreasing follows at once from the fact that, for all i,

$$\begin{aligned} e_i' C^{r+1} v &= e_i' C \,.\, C^r v \\ &\leqslant \max_{(i)} (e_i' C^r v), \end{aligned}$$

and $$\geqslant \min_{(i)} (e_i' C^r v).\dagger$$

A proof that $\max\limits_{(i)} (e_i' C^r v)$ and $\min\limits_{(i)} (e_i' C^r v)$ tend to the same limit as r tends to infinity may be modelled on the proof of Theorem III in (**1**).

† This step uses the fact that each row sum of C is equal to 1. The $\leqslant$, $\geqslant$ signs may be replaced by $<$, $>$ respectively except when $C^r v$ has all its elements equal. A similar remark applies to the inequalities in Theorem II of (**1**).

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A SIMPLIFYING TECHNIQUE IN THE SOLUTION OF A CLASS OF DIFFRACTION PROBLEMS

By D. S. JONES (*Manchester*)

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Summary

EXAMPLES are given which show that considerable simplification can be achieved in the formulation of diffraction problems which lead to Wiener–Hopf integral equations by taking the transform before applying the boundary conditions.

Introduction

In the theories of electromagnetism and small-amplitude sound waves a class of problems involving perfectly conducting or rigid parallel-plate or cylindrical structures can be solved by using Wiener–Hopf integral equations. These integral equations are derived by applying the boundary conditions when the field is expressed in terms of integrals of unknowns on various surfaces; in electromagnetism the unknowns are usually the current densities on the diffracting surfaces. The formation of the integral expressions often requires complicated functions, e.g. the Green's function of a cylinder used by Levine and Schwinger (**5**). It is the object of this note to show that these complications can be avoided, and that the Laplace transform of the integral equation can be obtained simply by taking the transform of the differential equation before applying the boundary conditions.†

The utility of the method is best demonstrated by examples, and I give four of them. The first example is the Sommerfeld problem for the semi-infinite plane; here the integral-equation formulation is particularly simple and the object of the example is to show that the new technique is no more complicated and that, moreover, it provides additional information. The second example is the diffraction of small-amplitude sound waves by a semi-infinite rigid cylinder; here the new technique is considerably simpler than the integral-equation formulation, which requires a lengthy and complicated analysis. The third example is the diffraction of a plane electromagnetic wave by a finite wave-guide; in

† The author and Mr. G. E. H. Reuter have also used this technique to solve problems involving other transforms, e.g. Mellin transforms when the electrified-disk problem is formulated in spherical polar coordinates.

this case the integral-equation formulation involves the kernel $|x+x'|$ as well as $|x-x'|$. It is shown that the new technique can still deal with this and remain simple. The fourth example is concerned with a vector field.

1. The Sommerfeld problem

We consider phenomena which are two-dimensional and depend only upon the Cartesian coordinates x and y. Let the semi-infinite plane be given by $x \geqslant 0$, $y = 0$ and let the plane wave $u_0 = e^{-ikx\cos\theta - iky\sin\theta}$ $(0 < \theta \leqslant \frac{1}{2}\pi)$ be the incident field. Let the total field be $u+u_0$, so that u is the solution of $\nabla^2 u + k^2 u = 0$ where $\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$ such that $u+u_0 = 0$ on the semi-infinite plane. (An example of the technique when the boundary condition is the vanishing of the normal derivative is given in § 2). The integral-equation method of solving the allied problem in sound waves is given by Baker and Copson (**1**).

We suppose that k has a small negative imaginary part so that

$$k = k_r - ik_i \quad (k_r > 0, k_i > 0).$$

Let

$$U(y) = \int_{-\infty}^{\infty} ue^{-sx}\, dx,$$

so that $U(y)$ is a function of s and y although the dependence on s is not explicitly indicated. Then U satisfies the equation

$$\partial^2 U/\partial y^2 + \kappa^2 U = 0, \tag{1}$$

where $\kappa^2 = s^2 + k^2$.

We now investigate the values of s for which (1) holds. As $x \to \infty$, $u = O(e^{-k_i x\cos\theta})$ either because there is a reflected wave $(y < 0)$ or because the incident plane wave is removed in the region of shadow $(y > 0)$. As $x \to -\infty$, u behaves like a diffracted wave and therefore $u = O(e^{-k_i|x|}/|x|^{\frac{1}{2}})$. Hence U is analytic in the strip $-k_i\cos\theta < \sigma < k_i$ $(s = \sigma + i\tau)$, and so (1) holds in this strip.

Define κ by choosing that branch of $\sqrt{(s^2+k^2)}$ which reduces to k when $s = 0$. Then, in the strip $-k_i\cos\theta < \sigma < k_i$,

$$U(y) = \begin{cases} A_1 e^{-i\kappa y} + B_1 e^{i\kappa y} & (y > 0), \\ A_2 e^{-i\kappa y} + B_2 e^{i\kappa y} & (y < 0), \end{cases}$$

where A_1, B_1, A_2, B_2 are functions of s only. There are two forms of U since $\partial u/\partial y$, and hence $\partial U/\partial y$, is discontinuous across $y = 0$. Since u is an outgoing wave as $|y| \to \infty$, we have, since $k_i > 0$, $u \to 0$ as $|y| \to \infty$. Hence $U \to 0$ as $|y| \to \infty$ and hence, since κ has a negative imaginary part for $-k_i < \sigma < k_i$, $B_1 = 0$, $A_2 = 0$. Since u is continuous across

$y = 0$, so is U and hence $A_1 = B_2 = A$ (say). Hence $U(y) = Ae^{-i\kappa|y|}$ (all y).

Write $U(y) = U_+(y) + U_-(y)$ where

$$U_+(y) = \int_0^\infty ue^{-sx}\,dx, \qquad U_-(y) = \int_{-\infty}^0 ue^{-sx}\,dx.$$

Then

$$U_+(0) + U_-(0) = A, \tag{2}$$

$$U'_+(+0) + U'_-(+0) = -i\kappa A, \tag{3}$$

$$U'_+(-0) + U'_-(-0) = i\kappa A, \tag{4}$$

where the primes indicate derivatives with respect to y.

Since $\partial u/\partial y$ is continuous in crossing $y = 0$, $x < 0$, we have

$$U'_-(+0) = U'_-(-0)$$

and hence, adding (3) and (4),

$$2U'_-(0) = -U'_+(+0) - U'_+(-0).$$

The right-hand side is analytic in $\sigma > -k_i \cos\theta$. The left-hand side is analytic in $\sigma < k_i$. Since the two sides have a strip in common, they are both equal to an integral function. It is easy to show, by considering the growths at infinity, that the integral function is zero. Hence

$$U'_-(0) = 0, \qquad U'_+(+0) = -U'_+(-0).$$

Since $U'_-(0) = 0$ for all s in the strip, it follows that $\partial u/\partial y = 0$ on $y = 0$, $x < 0$. This result is independent of the incident field and is valid, provided that the additional field has a suitable transform. It is usually deduced by the method of images.

Using the results of the previous paragraph in (2) we have

$$\begin{aligned} U'_+(+0)/\kappa &= -iU_+(0) - iU_-(0) \\ &= -iU_-(0) + i/(s + ik\cos\theta) \end{aligned}$$

since $u = -e^{-ikx\cos\theta}$ on $y = 0$, $x > 0$. Since $U_-(0)$, analytic in $\sigma < k_i$, is the transform of the field on $y = 0$, $x < 0$ and $U'_+(+0)$, analytic in $\sigma > -k_i \cos\theta$, is proportional to the transform of the current on the semi-infinite plane, this equation is the one that would be obtained by means of integral equations. We find without difficulty that

$$(s + ik\cos\theta)U'_+(+0) = i\sqrt{(-ik - ik\cos\theta)}\sqrt{(s + ik)}$$

and hence that

$$U(y) = \frac{-\sqrt{(-ik - ik\cos\theta)}}{(s + ik\cos\theta)\sqrt{(s - ik)}}\, e^{-i\kappa|y|}.$$

An inverse transformation immediately gives the field at any point:

this is a further advantage over the integral-equation method where the transform of the field has to be formed first.

2. The semi-infinite cylinder

We use cylindrical polar coordinates ρ, ϕ, z and let the semi-infinite cylinder be $\rho = a$, $z \geqslant 0$. We suppose that a is so small that only the fundamental mode $e^{\pm ikz}$ can propagate. Let the cylinder be excited internally by the sound wave $u_0 = e^{ikz}$ starting at $z = \infty$ and proceeding towards the origin. Let $u+u_0$ be the total field in $\rho \leqslant a$ and u the field in $\rho \geqslant a$. We require a solution of

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\phi^2}+\frac{\partial^2 u}{\partial z^2}+k^2u = 0$$

such that $\partial u/\partial\rho$ is continuous across $\rho = a$, such that $\partial u/\partial\rho = 0$ on $\rho = a$, $z \geqslant 0$ and such that, on $z \leqslant 0$,

$$[u]_{\rho=a+0} = [u+u_0]_{\rho=a-0}. \tag{5}$$

The integral-equation formulation has been given by Levine and Schwinger (**5**).

Let
$$U = \int_{-\infty}^{\infty} ue^{-sx}\,dx$$

so that, with our assumptions, U is analytic in the strip $-k_i < \sigma < k_i$ and

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial U}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 U}{\partial\phi^2}+\kappa^2U = 0.$$

Since u is single-valued in ϕ, is an outgoing wave as $\rho \to \infty$, and since U is finite at $\rho = 0$, we have

$$U(\rho) = \begin{cases} \sum_{n=-\infty}^{\infty} a_n H_n^{(2)}(\kappa\rho)e^{in\phi} & (\rho \geqslant a), \\ \sum_{n=-\infty}^{\infty} b_n J_n(\kappa\rho)e^{in\phi} & (\rho \leqslant a), \end{cases}$$

where J_n is the Bessel function of the nth order and $H_n^{(2)}$ is the Hankel function of the second kind and nth order. Since $\partial u/\partial\rho$ is continuous across $\rho = a$, we have

$$a_n H_n^{(2)\prime}(\kappa a) = b_n J_n'(\kappa a), \tag{6}$$

where primes indicate derivatives with respect to κa.

In a similar way to that used in § 1 we find

$$U_+(a+0)+U_-(a+0) = \sum_{n=-\infty}^{\infty} a_n H_n^{(2)}(\kappa a)e^{in\phi}, \tag{7}$$

$$U_+(a-0)+U_-(a-0) = \sum_{n=-\infty}^{\infty} b_n J_n(\kappa a)e^{in\phi}, \tag{8}$$

$$U'_+(a)+U'_-(a) = \sum_{n=-\infty}^{\infty} \kappa a_n H_n^{(2)\prime}(\kappa a)e^{in\phi}.$$

Write $V_+(a) = U_+(a+0)-U_+(a-0)$; then, subtracting (7) and (8) and using (5) and (6), we obtain

$$V_+(a)-1/(s-ik) = \sum_{n=-\infty}^{\infty} 2ia_n e^{in\phi}/\{\pi\kappa a J'_n(\kappa a)\},$$

since $$H_n^{(2)}(\kappa a)J'_n(\kappa a)-H_n^{(2)\prime}(\kappa a)J_n(\kappa a) = 2i/\pi\kappa a.$$

If we assume that $U = \sum_{n=-\infty}^{\infty} U^{(n)}e^{in\phi}$, where $U^{(n)}$ is independent of ϕ, we see that $U^{(n)} = 0$ $(n \neq 0)$ and that

$$U_-^{(0)\prime}(a) = \kappa a_0 H_0^{(2)\prime}(\kappa a),$$

$$V_+^{(0)}(a)-1/(s-ik) = 2ia_0/\{\pi\kappa a J'_0(\kappa a)\}$$

since $$U'_+(a) = 0.$$

Hence

$$V_+^{(0)}(a)-1/(s-ik) = 2iU_-^{(0)\prime}(a)/\{\pi a\kappa^2 J_1(\kappa a)H_1^{(2)}(\kappa a)\}.$$

This equation is closely related to equations (V.2) and (V.4) in the paper of Levine and Schwinger. The main difference is that this equation has a solution not involving an unknown constant which has to be determined by other methods [cf. equation (V.43)]. The solution to this equation is

$$\kappa^2 L_+(s)V_+^{(0)}(a) = (s+ik)L_+(s)-2ikL_+(ik),$$

where $L_+(s)/L_-(s) = J_1(\kappa a)H_1^{(2)}(\kappa a)$, $L_+(s)L_-(-s) = 1$. L_+ and L_- are similar to functions given by Levine and Schwinger and may easily be deduced from them. We can now find a_0 and hence the field everywhere.

3. The waveguide of finite length

The field is taken as two-dimensional with the electric vector in the direction of the z-axis. The plates of the waveguide occupy $-l \leqslant x \leqslant 0$, $y = 0$ and $-l \leqslant x \leqslant 0$, $y = d$ $(\pi < kd < 2\pi)$. Let the incident field be the plane wave $u_0 = \exp(-ikx\cos\theta-iky\sin\theta)$ $(0 \leqslant \theta \leqslant \frac{1}{2}\pi)$ and let $u+u_0$ be the total field. Then u is a solution of $\nabla^2 u+k^2 u = 0$ such

that $u+u_0 = 0$ on $-l \leqslant x \leqslant 0$, $y = 0$, $y = d$. The integral equation formulation is given by Jones (4).

As in § 1 we have

$$U(y) = \begin{cases} A_1 e^{-i\kappa y} & (y \geqslant d), \\ A_2 e^{-i\kappa y} + B_2 e^{i\kappa y} & (0 \leqslant y \leqslant d), \\ B_3 e^{i\kappa y} & (y \leqslant 0). \end{cases}$$

Since u is continuous, we have

$$A_2+B_2 = B_3, \qquad A_1 = A_2+B_2 e^{2i\kappa d}.$$

Hence, on writing

$$U = U_+ + U_- + U_1 = \left(\int_0^\infty + \int_{-\infty}^{-l} + \int_{-l}^{0} \right) u e^{-sx}\, dx,$$

it follows that

$$U_+(d)+U_-(d)+U_1(d) = A_2 e^{-i\kappa d} + B_2 e^{i\kappa d},$$

$$U_+(0)+U_-(0)+U_1(0) = A_2+B_2,$$

where $$(s+ik\cos\theta)U_1(d) = -e^{-ikd\sin\theta}(e^{sl+ikl\cos\theta}-1).$$

From the discontinuity of $\partial u/\partial y$ across the diffracting planes we deduce that

$$V_1(d) = U_1'(d+0) - U_1'(d-0) = -2i\kappa B_2 e^{i\kappa d},$$

$$V_1(0) = -2i\kappa A_2.$$

Eliminating A_2, B_2 we obtain

$$U_+(d)+U_-(d)+U_1(d) = \{V_1(0)e^{-i\kappa d}+V_1(d)\}/(-2i\kappa),$$

$$U_+(0)+U_-(0)+U_1(0) = \{V_1(0)+V_1(d)e^{-i\kappa d}\}/(-2i\kappa),$$

and the addition and subtraction of these equations gives

$$H_+ + H_- + H_1 = -(1/2i\kappa)(1+e^{-i\kappa d})\{V_1(0)+V_1(d)\}$$

$$= -\{V_1(0)+V_1(d)\}/K(s),$$

$$J_+ + J_- + J_1 = -(1/2i\kappa)(1-e^{-i\kappa d})\{V_1(d)-V_1(0)\},$$

where $H_+ = U_+(d)+U_+(0)$, $J_+ = U_+(d)-U_+(0)$ and similarly for the other quantities involved. $K(s)$ is defined in the same way as in the paper by Jones. Writing $K(s) = K_+(s)/K_-(s)$ and $K_+(s)K_-(-s) = -1$ [cf. equation (19) of (4)], we see that

$$(H_+ + H_- + H_1)K_+ = -\{V_1(0)+V_1(d)\}K_-. \tag{9}$$

Now, a use of Cauchy's theorem and the properties of H_-, K_+ at infinity give

$$H_-K_+(s) = -\frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty}\frac{H_-(w)K_+(w)}{w-s}\,dw+\frac{1}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty}\frac{H_-(w)K_+(w)}{w-s}\,dw, \tag{10}$$

where $-k_i < c_1 < \sigma < c_2 < k_i$. The first term of (10), by analytic continuation, is analytic in $\sigma > -k_i$ and the second in $\sigma < k_i$. Dealing similarly with the term $U_1(d)$ in H_1 we can split (9) into two parts, one of which is analytic in $\sigma > -k_i$ and the other in $\sigma < k_i$. Hence the two parts equal an integral function which is easily shown to be zero. Therefore

$$K_+(s)\left\{H_+ + \frac{1+e^{-ikd\sin\theta}}{s+ik\cos\theta}\right\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{K_+(w)}{w-s}\left\{H_- - \frac{e^{vl+ikl\cos\theta}(1+e^{-ikd\sin\theta})}{w+ik\cos\theta}\right\}dw$$

where $-k_i < c < k_i$ and $\Re(s-w) > 0$. A change of notation and slight manipulation show that this equation is equivalent to (22) of (4).

4. Three-dimensional diffraction by a semi-infinite plane

We consider the diffraction by the semi-infinite plane $x \geqslant 0$, $y = 0$, $-\infty < z < \infty$ of the harmonic plane wave whose electric intensity is given by

$$\mathbf{e}_0 e^{-\gamma z} = (\lambda, \mu, \nu)\exp(-\alpha x - \beta y - \gamma z)$$

where

$$\alpha = ik\sin\theta\cos\phi, \qquad \beta = ik\sin\theta\sin\phi, \qquad \gamma = ik\cos\theta,$$
$$\tfrac{1}{2}\pi \geqslant \theta > 0, \qquad \pi > \phi > 0,$$

and the time dependence $e^{i\omega t}$ is understood. The field depends upon z only through the factor $e^{-\gamma z}$. Let the total field be $(\mathbf{e}+\mathbf{e}_0)e^{-\gamma z}$, $(\mathbf{b}+\mathbf{b}_0)e^{-\gamma z}$, where $\mathbf{e}$, $\mathbf{b}$, related to the electric intensity and magnetic flux density respectively, are independent of z. The rectangular components of $\mathbf{e}$ satisfy

$$(\nabla^2+\gamma^2+k^2)\mathbf{e} = 0, \qquad \operatorname{div}(\mathbf{e}e^{-\gamma z}) = 0$$

and are such that the x- and z-components of $\mathbf{e}$ are zero on the semi-infinite plane. The magnetic flux density is determined by the equation

$$-i\omega\mathbf{b}e^{-\gamma z} = \operatorname{curl}(\mathbf{e}e^{-\gamma z}). \tag{11}$$

Integral-equation formulations have been given by Copson (2) and Jones (3).

Taking the transform with respect to x and denoting the transforms of $\mathbf{e}$, $\mathbf{b}$ by $\mathbf{E}$, $\mathbf{B}$, we find that $(\partial^2/\partial y^2+\kappa_1^2)\mathbf{E} = 0$ where $\kappa_1^2 = \kappa^2+\gamma^2$. Hence

$$\mathbf{E}(y) = \begin{cases} \mathbf{A}e^{-i\kappa_1 y} & (y \geqslant 0), \\ \mathbf{C}e^{i\kappa_1 y} & (y \leqslant 0) \end{cases}$$

where $\mathbf{A}$, $\mathbf{C}$ are vectors which depend only on s.

We indicate the components of vectors by the suffixes x, y, z. The continuity of e_x, e_z implies that $A_x = C_x$, $A_z = C_z$. Also the transform of $\operatorname{div}(\mathbf{e}e^{-\gamma z}) = 0$ gives

$$sA_x - i\kappa_1 A_y - \gamma A_z = 0,$$
$$sC_x + i\kappa_1 C_y - \gamma C_z = 0,$$

from which we deduce that

$$i\kappa_1(A_y - C_y) = 2sA_x - 2\gamma A_z. \tag{12}$$

$\mathbf{B}$ is determined by the transform of (11), which gives

$$-i\omega\mathbf{B}(y) = (s\mathbf{i} \mp i\kappa_1\mathbf{j} - \gamma\mathbf{k}) \wedge \mathbf{E}(y),$$

the upper or lower sign being taken according as $y \gtrless 0$.

Write $\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_-$, where

$$\mathbf{E}_+ = \int_0^\infty \mathbf{e}e^{-sx}\,dx, \qquad \mathbf{E}_- = \int_{-\infty}^0 \mathbf{e}e^{-sx}\,dx,$$

and then

$$E_{+x}(0) + E_{-x}(0) = A_x, \tag{13}$$

$$E_{+z}(0) + E_{-z}(0) = A_z, \tag{14}$$

$$4\pi k^2 \mathscr{J}_{+z} = -i\omega\{B_{+x}(+0) - B_{+x}(-0)\} = 2(s\gamma A_x + \kappa^2 A_z)/i\kappa_1, \tag{15}$$

$$-4\pi k^2 \mathscr{J}_{+x} = -i\omega\{B_{+z}(+0) - B_{+z}(-0)\}$$
$$= -2\{s\gamma A_z + (\gamma^2+k^2)A_x\}/i\kappa_1, \tag{16}$$

where we have used (12) in deriving (15) and (16). Since

$$E_{+x}(0) = -\lambda/(s+\alpha), \qquad E_{+z}(0) = -\nu/(s+\alpha),$$

we find from (13), (14), and (16) that

$$2\pi i k^2 \kappa_1 \mathscr{J}_{+x} + \frac{s\gamma\nu + \lambda(\gamma^2+k^2)}{(s+\alpha)} = s\gamma E_{-z}(0) + (s^2+k^2)E_{-x}(0),$$

which, apart from notational changes, is the equation immediately before (16) in the paper of Jones (3).

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ON THE PRODUCT OF TWO LINEAR FORMS, ONE HOMOGENEOUS AND ONE NON-HOMOGENEOUS

By P. KANAGASABAPATHY (*London*)

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1. LET X, Y be homogeneous linear forms in u, v of determinant 1, with real coefficients, and suppose that neither form represents zero for integers u, v not both zero. Let c be any real number, not zero. H. Davenport† has recently proved that there exist integers u, v, not both zero, such that

$$|(X+c)Y| < \frac{1}{4{\cdot}1}.$$

Further, by considering an example, he has shown that the result does not hold if the constant 4·1 is replaced by 5·06.

In this paper, I prove that the result is still true if the number 4·1 is replaced by 4·25, but is not true if it is replaced by 4·28536.... .

I wish to express my gratitude to Professor Davenport for his help and interest in the preparation of this paper.

2. To prove the first result, we need the following lemma.

LEMMA 1.‡ *Let X, Y be any two linear forms in u, v of determinant* 1, *neither of which represents zero for integers u, v not both zero; and let c be any real number other than zero. Then there exists an integral unimodular transformation from u, v to new variables u', v' which transforms $(X+c)Y$ into*

$$\pm\frac{1}{\theta+\phi}(u'+\theta v'-\alpha)(u'-\phi v'),$$

where

$$0 < \phi < 1, \qquad 1 \leqslant \alpha < \theta. \tag{1}$$

I now prove the following theorem.

THEOREM 1. *Let X, Y, and c be as already defined. Then there exist integers u, v, not both zero, such that*

$$|(X+c)Y| < \frac{1}{4{\cdot}25}.$$

† See H. Davenport, 'Note on a result of Chalk', *Quart. J. of Math.* (Oxford), above, pp. 130–8.

‡ Ibid. Lemma 2.

Proof. By Lemma 1, it is sufficient to show that, if θ, ϕ, α satisfy (1), then there exist integers u, v, not both zero, such that

$$|(u+\theta v-\alpha)(u-\phi v)| < \frac{1}{4\cdot 25}(\theta+\phi).$$

We assume the contrary, namely that

$$|(u+\theta v-\alpha)(u-\phi v)| \geqslant \frac{1}{4\cdot 25}(\theta+\phi) \tag{2}$$

for all integers u, v, not both zero, and deduce a contradiction.

We shall make frequent use of three cases of (2), namely

$$(1+\theta-\alpha)(1-\phi) \geqslant \frac{1}{4\cdot 25}(\theta+\phi), \tag{3}$$

$$(\theta-\alpha)\phi \geqslant \frac{1}{4\cdot 25}(\theta+\phi), \tag{4}$$

$$(1+2\theta-\alpha)|1-2\phi| \geqslant \frac{1}{4\cdot 25}(\theta+\phi). \tag{5}$$

Let $n < \alpha < n+1$, where n is a positive integer. Then from (2) we obtain

$$\alpha-n \geqslant \frac{1}{4\cdot 25n}(\theta+\phi) \tag{6}$$

and

$$n+1-\alpha \geqslant \frac{1}{4\cdot 25(n+1)}(\theta+\phi). \tag{7}$$

Hence, by addition,

$$\theta+\phi \leqslant \frac{n(n+1)}{2n+1}(4\cdot 25). \tag{8}$$

From (3) and (6) we obtain

$$1-\phi \geqslant \frac{\theta+\phi}{4\cdot 25(\theta+\phi-\phi-n+1)-(\theta+\phi)/n}. \tag{9}$$

Since the right-hand side decreases as $\theta+\phi$ increases, we can replace $\theta+\phi$ by its upper bound from (8), obtaining

$$1-\phi \geqslant \frac{n(n+1)}{4\cdot 25n(n+1)-(2n+1)\phi-2n^2}. \tag{10}$$

Similarly, from (5), (6), and (8) we obtain

$$|2\phi-1| \geqslant \frac{n(n+1)}{8\cdot 5n(n+1)-2(2n+1)\phi-2n^2}; \tag{11}$$

and from (4), (6), (8) we obtain

$$\phi \geqslant \frac{n(n+1)}{4\cdot 25n(n+1)-(2n+1)\phi-2n^2-2n-1}. \tag{12}$$

Case 1. Suppose first that $\phi < \frac{1}{2}$. Combining (11) and (12), we get

$$1 \geqslant \frac{n(n+1)}{8\cdot5n(n+1)-2(2n+1)\phi-2n^2}+ \\ +\frac{2n(n+1)}{4\cdot25n(n+1)-(2n+1)\phi-2n^2-2n-1}.$$

Since the right-hand side increases with ϕ, and $\phi > 0$, this implies

$$1 > \frac{1}{8\cdot5-2n/(n+1)}+\frac{2}{2\cdot25-1/n(n+1)}$$

$$> \frac{1}{7\cdot5}+\frac{2}{2\cdot25}, \quad \text{for all } n,$$

$$> 0\cdot133+0\cdot888,$$

and we have a contradiction.

Case 2. Suppose next that $\phi > \frac{1}{2}$. Then, by (11),

$$2\phi-1 \geqslant \frac{n(n+1)}{8\cdot5n(n+1)-2(2n+1)\phi-2n^2}.$$

Since the right-hand side increases with ϕ, and $\phi > \frac{1}{2}$, we have

$$2\phi-1 > \frac{1}{6\cdot5-1/n(n+1)}$$

$$> \frac{1}{6\cdot5}, \quad \text{for all } n.$$

Hence $$\phi > 0\cdot5769. \tag{13}$$

Similarly from (10) we obtain

$$1-\phi > \frac{1}{2\cdot25+(2n-1)/2n(n+1)} \geqslant \frac{1}{2\cdot5}, \tag{14}$$

for all n. Hence $$\phi < 0\cdot6. \tag{15}$$

Further, by combining (4) and (6), we get

$$\theta-n \geqslant \frac{(n+\phi)(\theta+\phi)}{4\cdot25n\phi}.$$

Hence, writing $\theta-n$ as $(\theta+\phi)-(n+\phi)$, we obtain

$$\theta+\phi \geqslant \left\{\frac{1}{n+\phi}-\frac{1}{4\cdot25n\phi}\right\}^{-1}. \tag{16}$$

Again, if $\theta-\alpha > m$, then, by (2),

$$\theta-\alpha-m \geqslant \frac{\theta+\phi}{4\cdot25(m+\phi)}.$$

Combining this inequality with (6), we obtain

$$\theta-(m+n) \geqslant \frac{(\theta+\phi)(m+n+\phi)}{4{\cdot}25n(m+\phi)}.$$

Hence
$$\theta+\phi \geqslant \left\{\frac{1}{m+n+\phi}-\frac{1}{4{\cdot}25n(m+\phi)}\right\}^{-1}. \tag{17}$$

Similarly, if $\theta-\alpha < l$, using (2) and (7) we obtain

$$\theta+\phi \leqslant \left\{\frac{1}{l+n+1+\phi}+\frac{1}{4{\cdot}25(n+1)(l+\phi)}\right\}^{-1}. \tag{18}$$

We now consider the different values of n.

(i) *Let* $n = 1$. By (12) we have

$$\phi \geqslant \frac{2}{8{\cdot}5-3\phi-5} > \frac{2}{8{\cdot}5-\frac{3}{2}-5} = 1,$$

giving a contradiction.

(ii) *Let* $n = 2$. By (12) we have

$$\phi \geqslant \frac{6}{12{\cdot}5-5\phi} > \frac{6}{12{\cdot}5-2{\cdot}5} = 0{\cdot}6,$$

which contradicts (15).

(iii) *Let* $n = 3$. By (8) we have

$$\theta+\phi \leqslant \tfrac{12}{7}(4{\cdot}25) < 7{\cdot}29. \tag{19}$$

By (16) with $n = 3$,

$$\theta+\phi \geqslant \left\{\frac{1}{3+\phi}-\frac{1}{12{\cdot}75\phi}\right\}^{-1}.$$

Since the right-hand side decreases as ϕ increases when $\phi < 0{\cdot}6$, we have

$$\theta+\phi > \left\{\frac{1}{3{\cdot}6}-\frac{1}{7{\cdot}65}\right\}^{-1}.$$

Hence
$$\theta+\phi > 6{\cdot}798. \tag{20}$$

Suppose that $\theta-\alpha > 3$. By (17), with $n = 3$ and $m = 3$, we have

$$\theta+\phi \geqslant \left\{\frac{1}{6+\phi}-\frac{1}{12{\cdot}75(3+\phi)}\right\}^{-1}.$$

Since the right-hand side increases with ϕ and $\phi > 0{\cdot}5$, we have

$$\theta+\phi > \left\{\frac{1}{6{\cdot}5}-\frac{1}{(12{\cdot}75)(3{\cdot}5)}\right\}^{-1} > 7{\cdot}6,$$

which contradicts (19). Hence $\theta-\alpha < 3$.

By (18), with $n = 3$ and $l = 3$, we have

$$\theta+\phi \leqslant \left\{\frac{1}{7+\phi}+\frac{1}{17(3+\phi)}\right\}^{-1}.$$

Since the right-hand side increases with ϕ and $\phi < 0{\cdot}6$, we get

$$\theta+\phi < \left\{\frac{1}{7{\cdot}6}+\frac{1}{61{\cdot}2}\right\}^{-1} < 6{\cdot}762,$$

which contradicts (20).

(iv) *Let* $n = 4$. By (8) we have

$$\theta+\phi \leqslant \tfrac{20}{9}(4{\cdot}25) < 9{\cdot}445. \tag{21}$$

Suppose that $\theta-\alpha > 4$. Then, since $\phi > 0{\cdot}5$, we have, by (17),

$$\theta+\phi > \left\{\frac{1}{8{\cdot}5}-\frac{1}{17(4{\cdot}5)}\right\}^{-1} > 9{\cdot}55,$$

which contradicts (21). Hence $\theta-\alpha < 4$.

By (18) with $n = 4$, $l = 4$, and from the fact that $\phi < 0{\cdot}6$, we obtain

$$\theta+\phi < \left\{\frac{1}{9{\cdot}6}+\frac{1}{(21{\cdot}25)(4{\cdot}6)}\right\}^{-1} < 8{\cdot}743.$$

Now, by (9),

$$1-\phi \geqslant \frac{\theta+\phi}{4{\cdot}25(\theta+\phi-\phi-3)-\frac{1}{4}(\theta+\phi)}.$$

Since the right-hand side decreases as $\theta+\phi$ increases, and $\theta+\phi < 8{\cdot}743$, we have

$$1-\phi > \frac{8{\cdot}743}{4{\cdot}25(5{\cdot}743-\phi)-2{\cdot}185}.$$

Again, since the right-hand side decreases as ϕ decreases, and $\phi > 0{\cdot}576$ by (13), we have

$$1-\phi > \frac{8{\cdot}743}{(4{\cdot}25)(5{\cdot}167)-2{\cdot}185} > 0{\cdot}442.$$

Hence $\phi < 0{\cdot}558$, which contradicts (13).

(v) *Let* $n = 5$. By (8) we have

$$\theta+\phi \leqslant \tfrac{30}{11}(4{\cdot}25) < 11{\cdot}591. \tag{22}$$

Suppose that $\theta-\alpha > 5$. By (17), since $\phi > 0{\cdot}576$, we have

$$\theta+\phi > \left\{\frac{1}{10{\cdot}576}-\frac{1}{(21{\cdot}25)(5{\cdot}576)}\right\}^{-1} > 11{\cdot}61,$$

which contradicts (22). Hence $\theta-\alpha < 5$.

By (18) with $l = n = 5$ and the fact that $\phi < 0{\cdot}6$, we have

$$\theta+\phi < \left\{\frac{1}{11{\cdot}6}+\frac{1}{(25{\cdot}5)(5{\cdot}6)}\right\}^{-1} < 10{\cdot}73.$$

Considering (9), and using the fact that $\theta+\phi < 10{\cdot}73$ and $\phi > 0{\cdot}576$, we obtain

$$1-\phi > \frac{10{\cdot}73}{(4{\cdot}25)(6{\cdot}154)-2{\cdot}146} > 0{\cdot}445.$$

Hence $\phi < 0{\cdot}555$, which contradicts (13).

(vi) *Let* $n = 6$. By (10), we have

$$\begin{aligned} 1-\phi &\geqslant \frac{42}{106{\cdot}5-13\phi} \\ &> \frac{42}{106{\cdot}5-13(0{\cdot}5769)} \\ &> 0{\cdot}424. \end{aligned}$$

Hence $\phi < 0{\cdot}576$, which contradicts (13).

(vii) *Let* $n = 7$. By (10), we have

$$\begin{aligned} 1-\phi &\geqslant \frac{56}{140-15\phi} \\ &> \frac{56}{140-15(0{\cdot}5769)} \\ &> 0{\cdot}426. \end{aligned}$$

Hence $\phi < 0{\cdot}574$, which contradicts (13).

(viii) *Let* $n \geqslant 8$. By (14) we have

$$1-\phi > \frac{1}{2{\cdot}25+(2n-1)/2n(n+1)}.$$

Since the right-hand side increases with n, and $n \geqslant 8$, we have

$$1-\phi > \frac{1}{2{\cdot}25+15/144} > 0{\cdot}4246.$$

Hence $\phi < 0{\cdot}5754$, which contradicts (13).

The proof of Theorem 1 is now complete.

3. For the proof of the second result, we need a further lemma.

LEMMA 2.† *If k is any positive integer, and U, V are any odd integers, then*

$$|U^2+2kUV-V^2| \geqslant 2k.$$

† See H. Davenport, *Proc. K. Nederland. Akad. van Wet.* 49 (1946), 815–21, Lemma 3.

THEOREM 2. *Let*

$$X+c = u+\theta v-\alpha \quad \textit{and} \quad Y = u-\phi v,$$

where $\theta = 6+5\sqrt{2}$, $\alpha = \frac{1}{2}\theta = 3+\frac{5}{2}\sqrt{2}$, *and* $\phi = \frac{1}{3}\sqrt{3}$.

Then

$$|(u+\theta v-\alpha)(u-\phi v)| \geqslant (\theta+\phi)/m$$

for all integers u, v *not both zero, where*

$$m = \frac{\theta+\phi}{(1+\theta-\alpha)(1-\phi)} = 4{\cdot}28536\ldots.$$

Equality occurs if and only if $u = v = 1$.

Proof. Suppose the theorem not true. Then there exist integers u, v, not both zero, such that

$$|(X+c)Y| < \frac{1}{m}(\theta+\phi) = 3{\cdot}1848\ldots. \tag{23}$$

Now $$(X+c)-Y = (6+5\sqrt{2}+\tfrac{1}{3}\sqrt{3})v-\tfrac{1}{2}(6+5\sqrt{2}).$$

Hence $$|X+c|+|Y| \geqslant \tfrac{1}{2}(6+5\sqrt{2}) = 6{\cdot}53\ldots. \tag{24}$$

It follows from (23) and (24) that either $|X+c| < \frac{3}{5}$, or $|Y| < \frac{3}{5}$; for we have $\lambda(6{\cdot}53-\lambda) > 3{\cdot}18$ when $\frac{3}{5} < \lambda \leqslant \sqrt{3}{\cdot}18\ldots$.

We now consider two cases.

Case 1. Suppose that $|X+c| < \frac{3}{5}$. Let $X'+c'$ be the algebraic conjugate form to $X+c$, so that

$$X'+c' = u+(6-5\sqrt{2})v-\tfrac{1}{2}(6-5\sqrt{2}).$$

Since $$X+c = u-v+\tfrac{1}{2}+(7+5\sqrt{2})(v-\tfrac{1}{2}),$$

we have $$(X+c)(X'+c') = \tfrac{1}{4}(U^2+14UV-V^2),$$

where $$U = 2(u-v)+1, \qquad V = 2v-1.$$

Hence, by Lemma 2, we have

$$|(X+c)(X'+c')| \geqslant \tfrac{7}{2} \tag{25}$$

for all integers u, v.

Since $|X+c| < \frac{3}{5}$, we have

$$u = -(6+5\sqrt{2})v+\tfrac{1}{2}(6+5\sqrt{2})+\xi, \tag{26}$$

where $|\xi| < \frac{3}{5}$. Using (25), and substituting for u from (26), we obtain

$$|(X+c)Y| \geqslant \frac{7}{2}\left|\frac{Y}{X'+c'}\right|$$

$$= \frac{7}{2}\left|\frac{(6+5\sqrt{2}+\frac{1}{3}\sqrt{3})v-\frac{1}{2}(6+5\sqrt{2})-\xi}{10\sqrt{2}v-5\sqrt{2}-\xi}\right|$$

$$= \frac{7}{2}\left|\frac{6+5\sqrt{2}+\frac{1}{3}\sqrt{3}}{10\sqrt{2}} + \frac{5\sqrt{2}(\frac{1}{3}\sqrt{3})-(5\sqrt{2}-6-\frac{1}{3}\sqrt{3})\xi}{10\sqrt{2}(10\sqrt{2}v-5\sqrt{2}-\xi)}\right|$$

$$\geqslant \frac{7}{2}\left\{\frac{6+5\sqrt{2}+\frac{1}{3}\sqrt{3}}{10\sqrt{2}} - \frac{5\sqrt{2}(\frac{1}{3}\sqrt{3})+(5\sqrt{2}-6-\frac{1}{3}\sqrt{3})\frac{3}{5}}{10\sqrt{2}(5\sqrt{2}-\frac{3}{5})}\right\}$$

$$> \tfrac{7}{2}\{0{\cdot}965-0{\cdot}048\} > 3{\cdot}2,$$

contrary to (23).

Case 2. Suppose that $|Y| < \frac{3}{5}$. Let Y' be the algebraic conjugate form to Y, so that

$$Y' = u+\tfrac{1}{3}\sqrt{3}v.$$

Then obviously

$$|YY'| \geqslant \tfrac{1}{3} \tag{27}$$

for all integers u, v, not both zero.

Since $|Y| < \frac{3}{5}$, we have

$$u = \tfrac{1}{3}\sqrt{3}v+\eta, \tag{28}$$

where $|\eta| < \frac{3}{5}$. Using (27), and substituting for u from (28), we obtain

$$|(X+c)Y| \geqslant \frac{1}{3}\left|\frac{X+c}{Y'}\right|$$

$$= \frac{1}{3}\left|\frac{(6+5\sqrt{2}+\frac{1}{3}\sqrt{3})v-\frac{1}{2}(6+5\sqrt{2})+\eta}{\frac{2}{3}\sqrt{3}v+\eta}\right|.$$

If $v > 0$, this expression is least when η is greatest. In this case, (23) implies

$$\frac{1}{3}\left\{\frac{(6+5\sqrt{2}+\frac{1}{3}\sqrt{3})v-\frac{1}{2}(6+5\sqrt{2})+\frac{3}{5}}{\frac{2}{3}\sqrt{3}v+\frac{3}{5}}\right\} < 3{\cdot}185,$$

whence $v \leqslant 4$.

If $v = -w$, where $w \geqslant 0$, the expression is least when η is least, and (23) implies

$$\frac{1}{3}\left\{\frac{(6+5\sqrt{2}+\frac{1}{3}\sqrt{3})w+\frac{1}{2}(6+5\sqrt{2})+\frac{3}{5}}{\frac{2}{3}\sqrt{3}w+\frac{3}{5}}\right\} < 3{\cdot}185,$$

which is false for all $w \geqslant 1$. Hence $v = 0$, which contradicts $|Y| < \frac{3}{5}$.

The remaining possibilities are tabulated below.

u	v	$\lvert(X+c)Y\rvert$
2	4	$(47{\cdot}7\ldots)(0{\cdot}30\ldots) > 14$
2	3	$(34{\cdot}6\ldots)(0{\cdot}26\ldots) > 8$
1	2	$(20{\cdot}60\ldots)(0{\cdot}1547\ldots) > 3{\cdot}187$
0	1	$(6{\cdot}53\ldots)(0{\cdot}577\ldots) > 3{\cdot}7$
1	1	$3{\cdot}1848\ldots$

Since $|(X+c)Y| \geqslant 3{\cdot}1848...$, with equality if and only if $u = v = 1$, Theorem 2 is established.

4. The constant $4{\cdot}28536...$ in Theorem 2 can be improved slightly by using the same values of θ and α but defining ϕ by

$$\phi = \tfrac{1}{73}(28+10\sqrt{2}).$$

Then the conclusion holds with $m = 4{\cdot}28471...$, and equality now occurs if and only if $u = 1$, $v = 1$ or $u = 1$, $v = 2$. The constant $4{\cdot}25$ in Theorem 1 can be improved to $4{\cdot}2575$ by more accurate calculations.

INTEGRALS REPRESENTING GENERAL HYPERGEOMETRIC TRANSFORMATIONS

By L. J. SLATER (*Cambridge*)

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1. Introduction

In a recent paper (**1**), D. B. Sears gave a transformation connecting $M+N+1$ general hypergeometric series of order $M+N+1$. His proof, which was based on Hill's identity for terminating series, used double induction in M and N, and, from this result, Sears deduced several general transformations of well-poised hypergeometric series. In a later paper (**2**), he gave the basic analogues of his theorems. In my previous paper (**3**), I deduced the corresponding results for bilateral series, of ordinary and basic types.

In the first part of this paper, I give a short proof, based on an integral of Barnes's type, of Sears's general theorem, and I state the corresponding integrals from which the general bilateral, the well-poised, and the well-poised bilateral transformations can be deduced. In the second part of the paper, I use a basic analogue of this Barnes integral to prove Sears's general transformation for basic series, and I deduce the basic analogues of the previous integrals. These now lead to general transformations between bilateral, well-poised, and well-poised bilateral basic series.

2. Notation

In the usual notation, let

$$(a)_n = a(a+1)(a+2)\ldots(a+n-1),$$

$$(a)_0 = 1, \qquad (a)_{-n} = (-1)^n/(1-a)_n,$$

$${}_MF_{M-1}\left[\begin{matrix}a_1, a_2, \ldots, a_M;\\ b_1, b_2, \ldots, b_{M-1};\end{matrix} x\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \ldots (a_M)_n\, x^n}{n!(b_1)_n (b_2)_n \ldots (b_{M-1})_n}.$$

This will be curtailed further to

$${}_MF_{M-1}[(a); (b); x],$$

when there is no danger of ambiguity. Also

$$\Gamma\left[\begin{matrix}a_1, a_2, \ldots, a_M\\ b_1, b_2, \ldots, b_N\end{matrix}\right] = \frac{\Gamma(a_1)\,\Gamma(a_2)\ldots\Gamma(a_M)}{\Gamma(b_1)\,\Gamma(b_2)\ldots\Gamma(b_N)}.$$

Similarly, this will be contracted to

$$\Gamma[(a); (b)],$$

and, where necessary, the number of each kind of parameter will be stated. $\sum_N$ will denote the summation from $r = 1$ to $r = N$.

3. The general theorem

A short proof of the first part of the general theorem will now be given. Consider the integral of Barnes's type,

$$2\pi i I_R = \int \Gamma\begin{bmatrix}(a+s),(1-B-s),-s\\(b+s),(1-A-s)\end{bmatrix} ds,$$

where there are M of the parameters a and b, $N+1$ of the parameters A, and N of the parameters B. I_R is taken in a clockwise direction round the usual contour consisting of a large semicircle, of centre O and radius R, to the right of the imaginary axis, and that part of the imaginary axis from $-iR$ to $+iR$, indented where necessary to ensure that all the poles in increasing sequences, that is $s = 1-B_1+n, 1-B_2+n,..., 1-B_N+n$, and n, for $n = 0, 1, 2,...$, lie to the right of the indented imaginary axis, and all the poles in decreasing sequences, that is

$$s = -a_1-n, -a_2-n,..., -a_M-n, \quad \text{for } n = 0, 1, 2,...,$$

lie to the left of this part of the contour.

On the semicircle, $s = Re^{i\theta}$,

and the integrand is $O(R^{-k})$,

where $k = \mathrm{re}[\sum (b+B-a-A)]$.

Let $R \to \infty$. Then, if $k > 0$, the integral round the semicircle tends to zero, and the integral along the imaginary axis exists, and is equal to the sum of all the residues of the poles in increasing sequences, that is

$$I = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \Gamma\begin{bmatrix}(a),(1-B),-s\\(b),(1-A),\end{bmatrix} ds$$

$$= \Gamma\begin{bmatrix}(a),(1-B)\\(b),(1-A)\end{bmatrix}{}_{M+N+1}F_{M+N}[(a),(A);(b),(B);1]+$$

$$+ \sum_N \Gamma\begin{bmatrix}(1+a-B_r),(B_r-B)',B_r-1\\(1+b-B_r),(B_r-A)\end{bmatrix}\times$$

$$\times {}_{M+N+1}F_{M+N}\begin{bmatrix}(1+c-B_r),(1+A-B_r);1\\2-B_r,(1+b-B_r),(1+B-B_r)\end{bmatrix} \quad (1)$$

where the dash denotes the omission of the vanishing factor B_r-B_r.

Similarly, by considering the integral taken clockwise round a semicircle to the left of the imaginary axis, we have

$$I = \sum_{M} \Gamma\begin{bmatrix} a_r, (a-a_r)', (1+a_r-B) \\ (1+a_r-A), (b-a_r) \end{bmatrix} \times$$
$$\times {}_{M+N+1}F_{M+N}\begin{bmatrix} a_r, (1+a_r-b), (1+a_r-B); \\ (1+a_r-a), (1+a_r-A) \end{bmatrix} 1 \quad \text{for } M+N \geqslant 2. \quad (2)$$

On equating these two results we have the first part of Sears's general theorem. As Sears shows, the second and third parts of his theorem follow from this result.

More generally, we can show from the integral

$$\int \Gamma[(a+s), (1-B-s), -s; (b+s), (1-A-s)]z^s \, ds, \quad (3)$$

taken round a large circle of radius R, where, again, there are M of the a and b parameters, $N+1$ of the A parameters, and N of the B parameters, that, if $|z| = 1$, and

$$I(z) = \Gamma[(a), (1-B); (b), (1-A)]_{M+N+1}F_{M+N}[(a), (A); (b), (B); z] +$$
$$+ \sum_{N} \Gamma[(1+a-B_r), (B_r-B)', B_r-1; (1+b-B_r), (B_r-A)] \times$$
$$\times {}_{M+N+1}F_{M+N}\begin{bmatrix} (1+a-B_r), (1+A-B_r); \\ 2-B_r, (1+b-B_r), (1+B-B_r); \end{bmatrix} z, \quad (4)$$

then

$$I(z) = \sum_{M} \Gamma[a_r, (a-a_r)', (1+a_r-B); (1+a_r-A), (b-a_r)] \times$$
$$\times {}_{M+N+1}F_{M+N}\begin{bmatrix} a_r, (1+a_r-b), (1+a_r-B); \\ (1+a_r-a), (1+a_r-A); \end{bmatrix} 1/z. \quad (5)$$

4. Well-poised results

Similarly, we can deduce the results for well-poised series. Thus the integral

$$\int_{-i\infty}^{i\infty} \Gamma[(a+s), (a-a_1-s); (1+a_1-A+s), (1-A-s)]\exp(2\pi i s) \, ds, \quad (6)$$

where there are $M+1$ of the a parameters and $M-1$ of the A parameters, provided that

$$\mathrm{re}[Ma_1+M-1-\textstyle\sum (a+A)] > 0,$$

leads to the result (1) (11.11), which expresses a ${}_{2M}F_{2M-1}[1]$ series, well-poised in a_1, in terms of M other ${}_{2M}F_{2M-1}[1]$ series, well-poised in $2a_r-a_1$.

Further, the integral

$$\int_{-i\infty}^{i\infty} \Gamma[(a+s), (a-a_1-s); (1+a_1-A+s), (1-A-s)]\exp(\pi i s) \, ds, \quad (7)$$

where there are $M+1$ of the a parameters and $M-1$ of the A parameters, and

$$\mathrm{re}[2Ma_1+2M-1-2\sum(a+A)]>0,$$

leads to the result (1) (11.12), which expresses a ${}_{2M}F_{2M-1}[-1]$ series, well-poised in a_1, in terms of M other ${}_{2M}F_{2M-1}[-1]$ series, well-poised in $2a_r-a_1$. The integral

$$\int_{-i\infty}^{i\infty}\Gamma[(a+s),(a-a_1-s);(1+a_1-A+s),(1-A-s)]\exp(\pi is)\,ds, \quad (8)$$

where there are $M+1$ of the a parameters and M of the A parameters, and

$$\mathrm{re}[(2M+1)a_1+2M-2\sum(a+A)]>0,$$

leads to (1) (11.13), which expresses a ${}_{2M+1}F_{2M}[1]$ series, well-poised in a_1, in terms of M other ${}_{2M+1}F_{2M}[1]$ series, well-poised in $2a_r-a_1$.

Finally, the integral

$$\int_{-i\infty}^{i\infty}\Gamma[(a+s),(a-a_1-s);(1+a_1-A+s),(1-A-s)]\exp(2\pi is)\,ds, \quad (9)$$

where there are $M+2$ of the a parameters and $M-1$ of the A parameters, and

$$\mathrm{re}[(2M+1)a_1+2M-1-2\sum(a+A)]>0,$$

leads to the result (1) (11.14), which expresses a ${}_{2M+1}F_{2M}[-1]$ series, well-poised in a_1, in terms of $M+1$ other ${}_{2M+1}F_{2M}[-1]$ series, well-poised in $2a_r-a_1$.

5. The bilateral theorem

From the integral

$$2\pi iI=\int_{-i\infty}^{i\infty}\Gamma[(a+s),(1-a-s);(b+s),(1-c-s)]\,ds, \quad (10)$$

where there are M each of the a, b, c parameters, taken along a similar path, we have

$$I=\sum_M\Gamma\begin{bmatrix}(1+a-a_r),(a_r-a)'\\(1+b-a_r),(a_r-c)\end{bmatrix}{}_{M+1}F_M[1,(1+c-a_r);(1+b-a_r);1]$$

$$=\sum_M\Gamma\begin{bmatrix}(a-a_r)',(1+a_r-a)\\(b-a_r),(1+a_r-c)\end{bmatrix}{}_{M+1}F_M[1,(1+a_r-b);(1+a_r-c);1], \quad (11)$$

provided that $\mathrm{re}[\sum(b-c)]>0$,

from which follows the general theorem for bilateral series (3) (10), which expresses an ${}_MH_M[1]$ series in terms of M other ${}_MH_M[1]$ series.

6. Well-poised bilateral integrals

In the same way, we can deduce the general transformations for well-poised bilateral series. From the integral

$$\int_{-i\infty}^{i\infty} \Gamma\begin{bmatrix}(a+s),(1+a_1-a+s),(1-a-s),(a-a_1-s)\\(1+a_1-c+s),(1-c-s)\end{bmatrix}\exp(\pi i s)\,ds \quad (12)$$

where there are $N+1$ of the a parameters and $2N$ of the c parameters, and

$$\mathrm{re}[2N-1+2Na_1-2\sum c]>0,$$

we deduce (**3**) (11), which expresses a ${}_{2N}H_{2N}[-1]$ series, well-poised in a_1, in terms of N other ${}_{2N}H_{2N}[-1]$ series, well-poised in $2a_r-a_1$.

From the integral

$$\int_{-i\infty}^{i\infty} \Gamma\begin{bmatrix}(a+s),(1+a_1-a+s),(1-a-s),(a-a_1-s)\\(1+a_1-c+s),(1-c-s)\end{bmatrix}\exp(2\pi i s)\,ds \quad (13)$$

with the same numbers of parameters, provided that

$$\mathrm{re}[N-1+Na_1-\sum c]>0,$$

we deduce (**3**) (12), which expresses a ${}_{2N}H_{2N}[1]$ series, well-poised in a_1, in terms of $N+1$ other ${}_{2N}H_{2N}[1]$ series, well-poised in $2a_r-a_1$.

From the integral

$$\int_{-i\infty}^{i\infty} \Gamma\begin{bmatrix}(a+s),(1+a_1-a+s),(1-a-s),(a-a_1-s)\\(1+a_1-c+s),(1-c-s)\end{bmatrix}\exp(\pi i s)\,ds, \quad (14)$$

where there are $N+1$ of the a parameters and $2N-1$ of the c parameters, and

$$\mathrm{re}[2N-4+(2N-1)a_1-2\sum c]>0,$$

we deduce (**3**) (13), which expresses a ${}_{2N-1}H_{2N-1}[1]$ series, well-poised in a_1, in terms of N other ${}_{2N-1}H_{2N-1}[1]$ series, well-poised in $2a_r-a_1$.

Finally, from the integral

$$\int_{-i\infty}^{i\infty} \Gamma\begin{bmatrix}(a+s),(1+a_1-a+s),(1-a-s),(a-a_1-s)\\(1+a_1-c+s),(1-c-s)\end{bmatrix}\exp(2\pi i s)\,ds, \quad (15)$$

with the same numbers of parameters, provided that

$$\mathrm{re}[2N-3+(2N-1)a_1-2\sum c]>0,$$

we deduce (**3**) (14), which expresses a ${}_{2N-1}H_{2N-1}[-1]$ series well-poised in a_1, in terms of N other ${}_{2N-1}H_{2N-1}[-1]$ series, well-poised in $2a_r-a_1$.

In particular, putting $N = 3$, $a_1 = a$, $a_2 = c_1 = 1+\frac{1}{2}a$, $a_3 = c_2 = b$, $a_4 = c_3 = c$, in the integral (14) above, we have the integral

$$\int_{-i\infty}^{i\infty} \Gamma\begin{bmatrix} a+s, 1+\frac{1}{2}a+s, b+s, c+s, 1+s, \\ 1-a-s, 1-\frac{1}{2}a-s, b-a-s, c-a-s, -s \\ 1+a-d+s, 1+a-e+s, 1-d-s, 1-e-s \end{bmatrix} \exp(\pi i s)\, ds \quad (16)$$

which leads in the same way, to the summation theorem for the well-poised bilateral ${}_5H_5(1)$ series with special first parameter, given originally by Dougall.

7. Basic notation

Next I shall consider the analogues of these results for basic series. Using F. H. Jackson's notation, we write,

$$[a]_n = (1-q^a)(1-q^{a+1})(1-q^{a+2})\ldots(1-q^{a+n-1}),$$
$$[a]_0 = 1, \qquad [a]_{-n} = (-1)^n q^{\frac{1}{2}n(n+1)}/q^{an}[1-a]_n,$$

and also

$${}_M\Phi_{M-1}\begin{bmatrix} a_1, a_2, \ldots, a_M; \\ b_1, b_2, \ldots, b_{M-1}; \end{bmatrix} x = \sum_{n=0}^{\infty} \frac{[a_1]_n[a_2]_n\ldots[a_M]_n\, q^{xn}}{[b_1]_n[b_2]_n\ldots[b_{M-1}]_n[1]_n}.$$

This will be contracted, as before, to

$${}_M\Phi_{M-1}[(a); (b); x].$$

Similarly,

$$\Pi_N\begin{bmatrix} a_1, a_2, \ldots, a_M \\ b_1, b_2, \ldots, b_N \end{bmatrix} = \prod_{n=0}^{N} \frac{(1-q^{a_1+n})(1-q^{a_2+n})\ldots(1-q^{a_M+n})}{(1-q^{b_1+n})(1-q^{b_2+n})\ldots(1-q^{b_N+n})},$$

and this will be contracted to $\Pi_N[(a); (b)]$, where the number of each kind of parameter will be stated, when necessary; Π will be written for Π_∞.

8. The general basic theorem

We consider first an integral which leads to the general basic theorem (2) (4.2). Suppose that a, b, c, d, e, and x are complex numbers such that none of the members of the sequences

$$-a-n, -b-n, 1-e+n, \text{ and } n, \quad \text{for } n = 0, 1, 2, \ldots,$$

coincide.

Let

$$P_N(s) = \Pi_N\begin{bmatrix} 1-c-s, d+s, 1-x-c+e+s, x+c-e-s \\ a+s, 1-e-s, b+s, -s \end{bmatrix}$$

where $0 < q < 1$, $q = e^{-t}$, $t > 0$. Then

$$q^{a+s} = e^{-(a+s)t},$$

and has the period $2\pi i/t$. Hence $1/\Pi[a+s]$ has poles at all the points

$$s = -a+n+2k\pi i/t,$$

in the s-plane.

We consider the integral

$$2\pi i I_N = t \int P_N(s)\, ds,$$

taken, in a clockwise direction, round the rectangular contour

$$A(-i\pi/t)\, B(i\pi/t)\, C(2N+i\pi/t)\, D(2N-i\pi/t),$$

for large enough N. AB is indented, if necessary, to ensure that all the points in the increasing sequences

$$1-e+n \text{ and } n, \quad \text{for } n = 0, 1, 2,...,$$

lie to the right of AB, and those in the decreasing sequences

$$-a-n \text{ and } -b-n, \quad \text{for } n = 0, 1, 2,...,$$

lie to the left of AB.

Now, by the periodicity of $P_N(s)$,

$$\int_{BC} P_N(s)\, ds + \int_{DA} P_N(s)\, ds = 0,$$

and, on CD, $s = 2N+ir$, where $-\pi/t \leqslant r \leqslant \pi/t$. Hence,

$$2\pi i I_N - t \int_{-i\pi/t}^{i\pi/t} P_N(s)\, ds = it \int_0^{\pi/t} [P_N(2N-ir) - P_N(2N+ir)]\, dr.$$

Now, if

$$P = \Pi_N \begin{bmatrix} a+2N+ir, b-2N-ir \\ c+2N+ir, d-2N-ir \end{bmatrix},$$

then

$$|P| \leqslant \Pi_N \begin{bmatrix} \mathrm{re}(a)+2N+\pi i/t, -\mathrm{re}(b)+2N+\pi i/t \\ \mathrm{re}(c)+2N, -\mathrm{re}(d)+2N \end{bmatrix} q^{\mathrm{re}(b-d)N};$$

that is,

$$|P| = O(q^{\mathrm{re}(b-d)N}).$$

Accordingly

$$\left| 2\pi i I_N - t \int_{-i\pi/t}^{i\pi/t} P_N(s)\, ds \right| < O(q^{N\,\mathrm{re}(x)}) t \int_0^{\pi/t} dr = O(q^{N\,\mathrm{re}(x)}),$$

which $\to 0$, as $N \to \infty$, if $\mathrm{re}(x) > 0$. Hence

$$I_N \to I = \frac{t}{2\pi i} \int_{-i\pi/t}^{i\pi/t} \Pi \begin{bmatrix} 1-c-s, d+s, 1-x-c+e+s, x+c-e-s \\ a+s, 1-e-s, b+s, -s \end{bmatrix} ds \tag{17}$$

as $N \to \infty$ for q real, $0 < q < 1$, under the above condition. But $P_N(s)$ has residues at all the points which fall within the strip $ABCD$ in the s-plane: that is,

$$\text{at } 1-e+n \text{ and } n \quad \text{for } n = 0, 1, 2,...,$$

or at $1-e+n+2\pi iK/t$ and $n+2\pi iK/t$, for some integer K, if the original points do not fall within this strip. Of each set of poles, only one can occur in any one strip.

From this we have

$$I_N = \Pi_N \begin{bmatrix} 1-c, d, 1-x-c+e, x+c-e \\ a, 1-e, b, 1 \end{bmatrix} \sum_{n=0}^{N} \frac{[a]_n[b]_n[c]_n q^{xn}}{[1]_n[d]_n[e]_n}[1+O(nq^N)]+$$

$$+\Pi_N \begin{bmatrix} e-c, 1+d-e, 2-x-c, x+c-1 \\ 1+a-e, 1+b-e, e-1, 1 \end{bmatrix} \times$$

$$\times \sum_{n=0}^{N} \frac{[1+a-e]_n[1+b-e]_n[1+c-e]_n q^{xn}}{[1]_n[1+d-e]_n[2-e]_n}[1+O(nq^N)].$$

When $N \to \infty$, using Tannery's theorem, we find

$$I = \Pi \begin{pmatrix} 1-c, d, 1-x-c+e, x+c-e \\ a, 1-e, b, 1 \end{pmatrix} {}_3\Phi_2 \begin{bmatrix} a, b, c; \\ d, e; \end{bmatrix} x \Big] +$$

$$+\Pi \begin{pmatrix} e-c, 1+d-e, 2-x-c, x+c-1 \\ 1+a-e, 1+b-e, e-1, 1 \end{pmatrix} {}_3\Phi_2 \begin{bmatrix} 1+a-e, 1+b-e, 1+c-e; \\ 1+d-e, 2-e; \end{bmatrix} x \Big]. \tag{18}$$

Also, by considering the poles of $P_N(s)$ in a similar strip to the left of AB, we have

$$I = \Pi \begin{pmatrix} 1+a-c, d-a, 1-x-a-c+e, x+c+a-e \\ b-a, 1+a-e, a, 1 \end{pmatrix} \times$$

$$\times {}_3\Phi_2 \begin{bmatrix} 1+a-d, 1+a-e, a; \\ 1+a-b, 1+a-c; \end{bmatrix} y-x \Big] +$$

$$+\text{a similar expression with } a \text{ and } b \text{ interchanged,} \tag{19}$$

where $y = 1+d+e-a-b-c$, and $\mathrm{re}(y) > 0$. By analytic continuation, these results (18) and (19) will hold now for $|q| < 1$. The basic result follows when we equate (18) and (19), under the conditions

$$\mathrm{re}(y) > \mathrm{re}(x) > 0.$$

If we write a for q^a, b for q^b, c for q^c, d for q^d, e for q^e, and x for q^x, then the above conditions become

$$|q| < 1, \qquad |de/abc| < |x| < 1,$$

and we have (2) (4.2) in the case $M = N = 1$. When $t \to 0$, then $q \to 1$, and (18) and (19) reduce formally to (1) and (2) with $M = N = 1$. Thus, (18) and (19) are exact basic analogues of (1) and (2).

Further, by considering the integral

$$2\pi i I_P = t \int \Pi_P(s)\, ds, \tag{20}$$

where $$\Pi_P(s) = \Pi_P\left[\begin{matrix}(1-A-s),(b+s),1-x-\alpha+s,x+\alpha-s\\(a+s),(1-B-s),-s\end{matrix}\right]$$

$$\alpha = \sum (A-B),$$

and there are $M+1$ of the a parameters, M of the b parameters, and N of the A and B parameters, taken round the same two strip contours, we can prove the basic theorem in the general case. For

$$\left|2\pi i I_P - t\int_{-i\pi/t}^{i\pi/t} \Pi_P(s)\, ds\right| = O[q^{P\,\mathrm{re}(x)}], \quad \text{to the right of } AB,$$

and

$$\left|2\pi i I_P - t\int_{-i\pi/t}^{i\pi/t} \Pi_P(s)\, ds\right| = O[q^{P\,\mathrm{re}(\beta-x)}], \quad \text{to the left of } AB,$$

where $$\beta = \sum (b+B-a-A).$$

Thus, if $$\mathrm{re}(\beta) > \mathrm{re}(x) > 0,$$

I_P exists and tends to I as $P \to \infty$, where

$$2\pi i I = t \int_{-i\pi/t}^{i\pi/t} \Pi\left[\begin{matrix}(1-A-s),(b+s),1-x-\alpha+s,x+\alpha-s\\(a+s),(1-B-s),-s\end{matrix}\right] ds. \tag{21}$$

From a consideration of the residues of the integrand within the same strip, $(\pm 2P - i\pi/t, i\pi/t, -i\pi/t, \pm 2P + i\pi/t)$, we deduce

$$I = \Pi\left[\begin{matrix}(1-A),(b),1-x-\alpha,x+\alpha\\(a),(1-B),1\end{matrix}\right]_{M+N+1}\Phi_{M+N}[(a),(A);(b),(B);x] +$$

$$+ \sum_N \Pi\left[\begin{matrix}(B_r-A),(1+b-B_r),2-x-\alpha-B_r,x+\alpha-1+B_r\\(1+a-B_r),(B_r-B)',2B_r-1,1\end{matrix}\right] \times$$

$$\times {}_{M+N+1}\Phi_{M+N}\left[\begin{matrix}(1+a-B_r),(1+A-B_r);\\(1+b-B_r),(1+B-B_r);\end{matrix} x\right]$$

$$= \sum_{M+1} \Pi\left[\begin{matrix}(1+a_r-A),(b-a_r),1-x-\alpha-a_r,x+\alpha+a_r\\(a-a_r)',(1+a_r-B),a_r,1\end{matrix}\right] \times$$

$$\times {}_{M+N+1}\Phi_{M+N}\left[\begin{matrix}(1+a_r-b),(1+a_r-B),a_r;\\(1+a_r-a),(1+a_r-A);\end{matrix} \beta-x\right], \tag{22}$$

where $$\alpha = \sum (A-B), \quad \text{and} \quad \beta = 1 + \sum (b+B-a-A).$$

When the notation is changed; that is, when a_1 is written for q^{a_1}, and so on, this becomes the general basic theorem **(2)** (4.3), and the convergence conditions become

$$|x| < 1, \quad \text{and} \quad |qb_1 \ldots b_{M+N}/a_1 \ldots a_{M+N+1}x| < 1.$$

9. Well-poised basic integrals

There are four integrals which lead, by similar methods, to the general relations **(2)** (7.2), (7.3), (7.4), and (7.5) between well-poised basic series. Of these integrals, the first is

$$\int_{-i\pi/t}^{i\pi/t} \Pi \left[\begin{matrix} (1-A-s), (1+a_1-A+s), 1+\frac{1}{2}a_1+s, 1-\frac{1}{2}a_1-s, \\ \qquad\qquad 1+\frac{1}{2}a_1+s+\pi i/t, 1-\frac{1}{2}a_1-s+\pi i/t \\ (a+s), (a-a_1-s) \end{matrix} \right] q^s \, ds, \tag{23}$$

where there are $M+1$ of the a parameters, and $M-1$ of the A parameters.

If in this integral, we put $2M+1$ for $2M$, and $A_M = \frac{1}{2}+\frac{1}{2}a_1$, we obtain the integral for **(2)** (7.3). Similarly, putting $M+1$ for M, and

$$A_M = \tfrac{1}{2}+\tfrac{1}{2}a_1,$$

we obtain the integral for **(2)** (7.4), and finally, putting $2M+1$ for $2M$, we obtain the integral for **(2)** (7.5).

10. Basic bilateral integrals

In the same way, from the integral

$$\int_{-i\pi/t}^{i\pi/t} \Pi \left[\begin{matrix} (1-c-s), (b+s), 1-x-\alpha+s, x+\alpha-s \\ (a+s), (1-a-s), 1+s, -s \end{matrix} \right] ds, \tag{24}$$

where

$$\alpha = \sum (c-a),$$

and there are M each of the a, b, c parameters, by considering the residues of the integral, taken round similar strip contours, we can deduce the general transformation for basic bilateral series **(3)** (4).

Finally, from the integral

$$\int_{-i\pi/t}^{i\pi/t} \Pi \left[\begin{matrix} (1-b-s), (1+a_1-b+s), 1+\frac{1}{2}a_1-s, 1-\frac{1}{2}a_1+s, \\ \qquad\qquad 1+\frac{1}{2}a_1+s+\pi i/t, 1-\frac{1}{2}a_1-s+\pi i/t \\ (a+s), (1-a-s), (1+a_1-a+s), (a-a_1-s) \end{matrix} \right] q^s \, ds, \tag{25}$$

where there are $M+1$ of the a parameters and M of the b parameters; and from three similar integrals, we can deduce the four corresponding transformations **(3)** (7), (8), and (9), for bilateral well-poised basic series of any order.

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INTEGRABILITY OF TRIGONOMETRIC SERIES (III)

By R. P. BOAS, Jr. (*Evanston, Ill.*)

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THIS note is mainly concerned with some integrability properties of trigonometric series with monotonic coefficients. They were suggested by the following two theorems about sine series and sine transforms which were proved by Zygmund (Theorem A for $\gamma = 1$) and by Sz.-Nagy in the general case (**4**).

THEOREM A. *If $f(x)$ is decreasing and bounded below in $(0, \pi)$, $xf(x) \in L$, and b_n are the Fourier sine coefficients of $f(x)$, then, for $0 < \gamma \leqslant 1$, $\sum n^{-\gamma}|b_n|$ converges if and only if $x^{\gamma-1}f(x) \in L$.*

THEOREM B. *If $f(x) \downarrow 0$ in $(0, \infty)$, $xf(x) \in L$ on every finite interval, and $b(x)$ is the Fourier sine transform of $f(x)$, then, for $0 < \gamma \leqslant 1$, $x^{-\gamma}|b(x)|$ belongs to $L(0,1)$ or $L(1,\infty)$ if and only if $x^{\gamma-1}f(x)$ belongs respectively to $L(1,\infty)$ or $L(0,1)$.*

There are corresponding theorems for cosine series and transforms, differing somewhat in detail; the most significant difference is the replacement of $f(x) \in L$ by $f(x)\log x \in L$ when $\gamma = 1$.

Since Theorem B can be read backwards as a theorem about $f(x)$ instead of about $b(x)$, it suggests a companion to Theorem A in which the coefficients are monotonic instead of the function. The main purpose of the present note is to prove such a theorem and the corresponding theorem for cosine series. These in turn suggest a theorem about the absolute convergence of a class of sine series with positive coefficients.

THEOREM 1. *If $f_n \downarrow 0$ and $b(x) = \sum_{n=1}^{\infty} f_n \sin nx$, then, for $0 \leqslant \gamma \leqslant 1$, $x^{-\gamma}b(x) \in L$ if and only if $\sum n^{\gamma-1}f_n$ converges.*

This is a precise analogue of Theorem A except that it is still true for $\gamma = 0$, where Theorem A obviously breaks down; the truth of Theorem 1 for $\gamma = 0$ suggests, however, that something of Theorem A should survive for $\gamma = 0$, and I give a result in this direction in Theorem 4 below. The case $\gamma = 0$ of Theorem 1 is in fact a theorem due to W. H. Young: *If $f_n \downarrow 0$, $b(x) \in L$ if and only if $\sum n^{-1}f_n < \infty$* [(**2**) 33; (**5**) 108].

The case $\gamma = 1$ of Theorem 1 is: *If $f_n \downarrow 0$, $b(x)/x \in L$ if and only if*

$\sum f_n < \infty$. The second part is a corollary of known results, since, if $b(x)/x \in L$, the conjugate Fourier series $\sum f_n \cos nx$ converges at $x = 0$ [(**2**) 48], i.e. $\sum f_n$ converges.

For cosine series, we have to separate the cases $0 < \gamma < 1$ and $\gamma = 1$.

THEOREM 2. *If* $f_n \downarrow 0$ *and* $a(x) = \sum_{n=1}^{\infty} f_n \cos nx$, *then, for* $0 < \gamma < 1$, $x^{-\gamma}a(x) \in L$ *if and only if* $\sum n^{\gamma-1} f_n$ *converges.*

For $\gamma = 1$, it is first of all necessary to relax the monotonic condition since, if $f_n \downarrow 0$ and $a(x) \not\equiv 0$, $a(0) \neq 0$ and $a(x)/x$ cannot be integrable. We then have the following theorem.

THEOREM 3. *If* $f_n \downarrow 0$ *for large* n, $\sum f_n$ *converges,*

$$a(x) = \tfrac{1}{2}f_0 + \sum_{n=1}^{\infty} f_n \cos nx, \quad \textit{and} \quad a(0) = 0,$$

then $a(x)/x \in L$ *if and only if* $\sum f_n \log n < \infty$.

The limiting case $\gamma = 0$ of Theorem 2 should involve a necessary and sufficient condition for $a(x)$ to be integrable when $f_n \downarrow 0$; the sufficient condition $\sum f_n/n < \infty$ is not necessary, and the problem of finding a necessary and sufficient condition appears to be difficult.

The final result is a partial extension of Theorem A to the case $\gamma = 0$; cf. the case $\gamma = 1$ of Theorem 1.

THEOREM 4. *If the Fourier series* $f(x) = \sum b_n \sin nx$ *does not change sign in a right-hand neighbourhood of* 0, *and if* $b_n \geqslant 0$, *then* $\sum b_n$ *converges if and only if* $f(x)/x \in L$.

To clarify the role of the hypotheses we note that, if $f(x)$ decreases throughout $(0, 2\pi)$, $b_n \geqslant 0$; but then $f(x)$ cannot be continuous everywhere; so the assumption that $f(x)$ is locally monotonic and $b_n \geqslant 0$ would be a natural one; we actually assume a little less. Furthermore, the condition $f(x)/x \in L$ is (under our hypotheses) weaker than the condition that $f'(x)\log x$ is integrable at 0, which by itself is less restrictive than Zygmund's condition [(**5**) 138] $f' \log^+|f'| \in L$ for absolute convergence; of course we have the strong additional hypothesis $b_n \geqslant 0$.

I shall actually prove somewhat more than Theorems 1, 2, and 3, by using only part of the hypotheses for each half of the theorems. We need the following simple lemma.

LEMMA. *If* $0 < \gamma \leqslant 1$, $f_n \downarrow 0$ *and* $\sum f_n n^{\gamma-1}$ *converges, then*

$$\sum |f_{n-1} - f_{n+1}| n^{\gamma} \tag{1}$$

converges.

This follows, by partial summation, as soon as we observe that, since $n^{\gamma-1}f_n \downarrow 0$ and $\sum f_n n^{\gamma-1}$ converges, $n^{\gamma}f_n \to 0$.

I first show that, if (1) converges and $f_n \to 0$, then $x^{-\gamma}b(x) \in L(0,\pi)$; this will prove the 'if' part of Theorem 1 for $0 < \gamma \leqslant 1$. We are actually concerned only with a neighbourhood of 0, since $b(x)$ is continuous except perhaps at 0 [(**5**) 3].

Write $c(x) = 1-\cos x$. Then by partial summation

$$\tfrac{1}{2}b(x) = \frac{1}{\sin x}\sum_{n=0}^{\infty} f_{n+1}[c\{(n+2)x\}-c(nx)]$$

$$= \frac{1}{\sin x}\Big[-f_2 c(x)+f_1 c(2x)+\sum_{n=3}^{\infty}(f_{n-1}-f_{n+1})c(nx)\Big].$$

Since $x^{-\gamma}c(jx)/\sin x$ $(j = 1, 2)$ is integrable and $\sin x \sim x$ as $x \to 0$, it is enough to show the existence of

$$\int_0^{\pi} x^{-\gamma-1}\sum_{n=3}^{\infty}|f_{n-1}-f_{n+1}|c(nx)\,dx$$

$$= \sum_{n=3}^{\infty}|f_{n-1}-f_{n+1}|\int_0^{\pi} x^{-\gamma-1}c(nx)\,dx$$

$$= \sum_{n=3}^{\infty}|f_{n-1}-f_{n+1}|n^{\gamma}\int_0^{n\pi} x^{-\gamma-1}c(x)\,dx$$

$$\leqslant \sum_{n=3}^{\infty}|f_{n-1}-f_{n+1}|n^{\gamma}\int_0^{\infty} x^{-\gamma-1}(1-\cos x)\,dx,$$

and the last expression is finite.

It is of interest to note that, when $\gamma = 1$, we also have $b(x)/(\pi-x) \in L$ in a neighbourhood of π (for $\gamma < 1$ the corresponding fact is, of course, trivial). For

$$b(\pi-x) = \sum_{n=1}^{\infty}(-1)^{n+1}f_n \sin nx,$$

and the series (1) is the same for $b(\pi-x)$ as for $b(x)$.

We prove the other half of Theorem 1 by proving more generally that, *if* $f_n \geqslant 0$ *and* $x^{-\gamma}b(x) \in L(0,\pi)$, $0 < \gamma \leqslant 1$, *then* $\sum f_n n^{\gamma-1} < \infty$. This part of Theorem 1 is thus really a theorem about Fourier series with positive, rather than monotonic, coefficients.

We have $b(x) \in L$ and

$$\tfrac{1}{2}\pi\sum_{n=1}^{m} n^{\gamma-1}f_n = \sum_{n=1}^{m} n^{\gamma-1}\int_0^{\pi} b(x)\sin nx\,dx = \int_0^{\pi} b(x)\Big\{\sum_{n=1}^{m} n^{\gamma-1}\sin nx\Big\}\,dx. \tag{2}$$

Now, if $0 < \gamma < 1$ and we write $q = [1/x]$, we have

$$\begin{aligned}\Big|\sum_{n=1}^{m} n^{\gamma-1}\sin nx\Big| &\leqslant \Big|\sum_{n=1}^{q}\Big| + \Big|\sum_{n=q+1}^{m}\Big| \\ &\leqslant \sum_{n=1}^{q} n^{\gamma-1} + q^{\gamma-1}\max_{m\geqslant p>q}\Big|\sum_{n=q+1}^{p}\sin nx\Big| \\ &\leqslant \sum_{n=1}^{q} n^{\gamma-1} + q^{\gamma-1}\max_{m\geqslant p>q}\left|\frac{\sin(p+\frac{1}{2})x - \sin(q+\frac{1}{2})x}{2\sin\frac{1}{2}x}\right| \\ &\leqslant Ax^{-\gamma},\end{aligned}$$

where A depends only on γ. If $\gamma = 1$, the same estimate follows from

$$\Big|\sum_{1}^{m}\sin nx\Big| = \left|\frac{\cos\frac{1}{2}x - \cos(m+\frac{1}{2})x}{2\sin\frac{1}{2}x}\right| \leqslant A/x.$$

Thus the last integral in (2) is at most

$$A\int_0^{\pi} x^{-\gamma}|b(x)|\,dx,$$

and so $\sum n^{\gamma-1}f_n$ converges.

The proof of Theorem 2 is similar to that of Theorem 1, and I give it in condensed form. First, if $f_n \to 0$ and (1) converges, $0 < \gamma < 1$, we have

$$\begin{aligned}\tfrac{1}{2}a(x) &= \frac{1}{\sin x}\sum_{n=0}^{\infty} f_{n+1}[\sin(n+2)x - \sin nx] \\ &= \frac{1}{\sin x}\Big[-f_2\sin x + f_1\sin 2x + \sum_{n=3}^{\infty}(f_{n-1}-f_{n+1})\sin nx\Big].\end{aligned}$$

Since $$\int_0^{\pi} x^{-\gamma-1}|\sin nx|\,dx = n^{\gamma}\int_0^{n\pi} x^{-\gamma-1}|\sin u|\,du = O(n^{\gamma}),$$

it follows that $a(x)x^{-\gamma} \in L$.

On the other hand, if $f_n \geqslant 0$ and $x^{-\gamma}a(x) \in L$, then $a(x) \in L$ and

$$\tfrac{1}{2}\pi\sum_{n=1}^{m} n^{\gamma-1}f_n = \sum_{n=1}^{m} n^{\gamma-1}\int_0^{\pi} a(x)\cos nx\,dx = \int_0^{\pi} a(x)\Big\{\sum_{n=1}^{m} n^{\gamma-1}\cos nx\Big\}\,dx. \tag{3}$$

But, with $q = [1/x]$,

$$\begin{aligned}\Big|\sum_{n=1}^{m} n^{\gamma-1}\cos nx\Big| &\leqslant \Big|\sum_{n=1}^{q}\Big| + \Big|\sum_{n=q+1}^{\infty}\Big| \\ &\leqslant \sum_{n=1}^{q} n^{\gamma-1} + q^{\gamma-1}\max_{m\geqslant p>q}\left|\frac{\sin(p+\frac{1}{2})x}{2\sin\frac{1}{2}x}\right| \\ &\leqslant Ax^{-\gamma}.\end{aligned}$$

Thus the last integral in (3) is bounded and $\sum n^{\gamma-1}f_n$ converges.

I now prove Theorem 3. If $\sum f_n$ converges and $a(0) = 0$, let $g_n = \sum_{k=n}^{\infty} f_k$, so that $\Delta g_n = f_n$ and $g_n \downarrow 0$. By partial summation we have

$$\begin{aligned} a(x) &= \sum_{n=1}^{\infty} g_n[\cos(n-1)x - \cos nx] \\ &= 2\sum_{n=2}^{\infty} g_n \sin(n-\tfrac{1}{2})x \sin \tfrac{1}{2}x \\ &= \sum_{n=2}^{\infty} g_n \sin nx \sin x - 2\sum_{n=2}^{\infty} g_n \cos nx \sin^2 \tfrac{1}{2}x. \end{aligned}$$

If $\sum f_n \log n = \sum |\Delta g_n| \log n < \infty$, then $\sum g_n \sin nx$ and $\sum g_n \cos nx$ are integrable (**3**), and so $a(x)/x \in L$.

On the other hand, by partial summation $\sum g_n \cos nx$ is at worst $O(1/x)$ as $x \to 0$, and so $\sum g_n \cos nx \sin \frac{1}{2}x$ is certainly integrable; hence, if $a(x)/x$ is integrable, so is $\sum g_n \sin nx$, and, since $g_n \downarrow 0$, this is true if and only if $\sum \Delta g_n . \log n$ converges [(**5**) 112], i.e. if $\sum f_n \log n$ converges.

Finally I prove Theorem 4. If $\sum b_n$ converges, the integral $\int_{\to 0}^{\pi} x^{-1} f(x)\, dx$ exists (**1**), but $f(x)$ has a fixed sign near 0, and so $x^{-1}f(x) \in L$. In the other direction, $x^{-1}f(x) \in L$ implies the convergence of the conjugate Fourier series $\sum b_n \cos nx$ at $x = 0$, i.e. the convergence of $\sum b_n$ [(**2**) 48].

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SOME INTEGRAL REPRESENTATIONS OF AN ANALYTIC FUNCTION

By F. B. PIDDUCK (*Keswick*)

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LET A, B be two circles of radii a, b in the complex plane with the origin as centre, where $a > b$. Let $f(z)$ be a single-valued function of z, analytic between A and B and containing no constant term in Laurent's expansion in direct and inverse powers of z. Write $f(z) = f_1(z)+f_2(1/z)$, where $f_1(z)$ is analytic inside A and $f_2(1/z)$ outside B. Let $\phi(u)$ be a function of a real or complex variable u, and $\Phi(z)$ a function of z such that

$$\int_p^q \phi(u)\Phi(uz)\,du = \frac{1}{1-z} \qquad \text{when } |z| < 1, \tag{1}$$

$\Phi(z)$ being analytic for all values of u included in the integral. Then inside A

$$\int_p^q \phi(u)\,du \int_A \frac{f_1(t)\Phi(uz/t)\,dt}{t} = 2\pi i f_1(z).$$

If $f_1(z)$ is replaced by $f(z)$ in the integral, the added terms contain inverse powers of t higher than the first and contribute nothing to the integral. Similar considerations apply to $f_2(1/z)$, using the variable $1/z$. If there is a constant term c_0 in Laurent's expansion, we can only replace $f_2(1/t)$ in its integral by $f(t)-c_0$. By addition we have for any function analytic in the annulus

$$\int_p^q \phi(u)\,du \int_A \frac{f(t)\Phi(uz/t)\,dt}{t} + \int_p^q \phi(u)\,du \int_B \frac{f(t)\Phi(ut/z)\,dt}{t} - \int_B \frac{f(t)\,dt}{t} = 2\pi i f(z). \tag{2}$$

If $\Phi(z)$ is an integral function of z, A can be replaced by any closed circuit C circulating positively in the annulus, and

$$2\pi i f(z) = \int_p^q \phi(u)\,du \int_C \frac{f(t)[\Phi(uz/t)+\Phi(ut/z)]\,dt}{t} - \int_C \frac{f(t)\,dt}{t} \tag{3}$$

at all points within the annulus.

Equation (3) can be applied in one case to discontinuous functions of a real variable, such as $f(z)$ may be on A or B. Writing the equation in the form $f(z) = \Omega f(t)$, let Ω be a real operator, by which we mean one that converts the real and imaginary parts of $f(t)$ into the real and

imaginary parts of $f(z)$. Take $f(z) = \psi(z)\log(z_0 - z)$, where $\psi(z)$ is analytic in a larger region than that considered. Take C to be a circle of radius just less than $|z_0|$, and the point z just inside C. Let $\psi(z)$ have the value 1 at the point P where Oz_0 meets C. Then the imaginary part of $f(z)$ changes by nearly $i\pi$ when z passes P, and its value at P is $\frac{1}{2}\{f(z+0)+f(z-0)\}$, where $z \pm 0$ are points indefinitely near to P on either side. Functions that become infinite in various degrees, while remaining integrable, are illustrated by the real part of $\psi(z)(z_0 - z)^{-\mu}$, where $0 < \mu < 1$; while $\psi(z)(z_0 - z)\log(z_0 - z)$ illustrates functions with a discontinuous derivate.

Equation (1) has a solution

$$\phi(u) = \exp(-u), \qquad \Phi(z) = \exp z,$$

when $p = 0$ and $q = \infty + ic$, where c is real. Let A, B be cut at the polar angle π, and the ends joined by two radii L, M at polar angles $-\pi+0$ and $\pi-0$, so that $f(z)$ is analytic in the curvilinear rectangle bounded by A, L, B, M. Change the variables from u to nu and from z to $\exp(iz/n)$, where n is real and positive. Writing c for c/n and $f(z)$ for $f\{\exp(iz/n)\}$, we have

$$2\pi f(z) = \int_0^{\infty+ic} du \int_C f(t)[\![\exp(nu[\exp\{i(z-t/n)\}-1]) + \\ +\exp(nu[\exp\{-i(z-t/n)\}-1])]\!]\, dt - n^{-1}\int_C f(t)\, dt. \quad (4)$$

Here A, B are segments of the lines $y = -n\log a$, $y = -n\log b$, and L, M segments of the lines $x = \pm n\pi$. Since L, M correspond to the same line in the original annulus, $f(z)$ must have equal values at points of equal ordinate on L and M, and C must begin and end at two such points. If we change the variables in (3) to inu and $z^{i/n}$, where $z^{i/n} = 1$ when $z = 1$, we have in a similar notation

$$2\pi i f(z) = \int_{c-i\infty}^{0} du \int_C f(t)(\exp[inu\{(z/t)^{i/n}-1\}]+\exp[inu\{(t/z)^{i/n}-1\}])\frac{dt}{t} + \\ +\frac{i}{n}\int_C \frac{f(t)\, dt}{t}, \quad (5)$$

where A, B are segments of straight lines through the origin making angles $-n\log a$, $-n\log b$ with the real axis, and L, M segments of circles of radii $\exp(-n\pi)$, $\exp n\pi$ with the origin as centre. In general $f(z)$ must have equal values at points of L and M of equal radius, and C must begin and end at two such points.

If $f(x)$ vanishes at infinity and $\int_{-\infty}^{\infty} f(x)\,dx$ is finite, equation (4) reduces to Fourier's integral theorem when $n \to \infty$. The first exponential in (5) tends to $(t/z)^u$, where $z^u = 1$ when $z = 1$. The second exponential tends to $(z/t)^u$. Write $-u$ for u in the second integral, in which we need not use the same c. Thus, if $f(z)$ is analytic in the space between two straight lines A, B through O, and satisfies certain other conditions,

$$2\pi i f(z) = \int_{c-i\infty}^{d+i\infty} \frac{du}{z^u} \int_C f(t) t^{u-1}\,dt,$$

where C is a straight line extending from O to infinity in the sector. Ω is a real operator if z and t are real and $c = d$. The condition that the last term of (5) shall tend to zero when $n \to \infty$ is satisfied most easily by making $\int_0^{\infty} f(x)\,dx/x$ finite. If we put a logarithmic singularity z_0 of polar angle π just outside the contour C of equation (3), we can relax the restriction that $f(z)$ shall have equal values at the points where C meets L and M. Thus $f(x)$ need not be zero at the origin in this proof of Mellin's theorem; and similarly $f'(x)$ need not be zero.

$$\text{If } \Phi(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad \frac{1}{a_n} = \int_p^q \phi(u) u^n\,du \tag{6}$$

from equation (1). In particular, if

$$p = 0, \quad q = \infty, \quad \text{and} \quad \phi(u) = u^{\alpha-1}\exp(-u),$$

where $\alpha > 0$, $\Phi(z) = {}_1F_1(1;\alpha;z)/\Gamma(\alpha)$. When u is large,

$$\phi(u)\Phi(uz) \approx \exp\{-(1-z)u\}.$$

The integral (1) converges when $R(z) < 1$. In this case, therefore, $|z/t|$ need not be less than 1 in the first integral of equation (3), if only $R(z/t) < 1$. Thus, if we draw Borel's polygon for the singularities of $f(z)$ outside A, and circles on OQ as diameter for the singularities Q inside B, (3) holds for all points between the polygon and the circles. If $p = 0$, $q > 0$, and $\phi(u) = u^{\alpha-1}(1-u/q)^{\beta-1}$ in that range, where α and β are positive,

$$\Phi(z) = \frac{\Gamma(\alpha+\beta)\,{}_2F_1(1, \alpha+\beta; \alpha; z/q)}{q^{\alpha}\Gamma(\alpha)\Gamma(\beta)} \tag{7}$$

when $|z| < 1$, reducing to the above when $\beta = q+1$ and $q \to \infty$. Otherwise $\Phi(z)$ is not an integral function of z, and equation (2) must be used

instead of (3). If $p = 0$ and $q = \infty$, another solution is

$$\phi(u) = u^{\alpha-1}K_\nu(u), \qquad \Phi(z) = \sum_{n=0}^{\infty} \frac{z^n}{2^{\alpha-2n}\Gamma(\frac{1}{2}\alpha+\frac{1}{2}\nu+\frac{1}{2}n)\Gamma(\frac{1}{2}\alpha-\frac{1}{2}\nu+\frac{1}{2}n)} \qquad (\alpha > \nu > 0).$$

Let C be a closed curve passing through the point z and through no singularity of $f(z)$, and H an operator defined by the equation

$$Hf(z) = \frac{1}{\pi i} P \int_C \frac{f(t)\,dt}{t-z}, \tag{8}$$

where P denotes the principal value of the integral; that is, a length ϵ is removed from C on either side of z, and ϵ made indefinitely small. Let the only singularities of $f(z)$ inside C be poles (of any order) at a finite number of points z_r, the principal part of $f(z)$ in the vicinity being $\sum_n a_n/(z-z_r)^n$. If

$$\frac{a_n}{(t-z)(t-z_r)^n} \equiv \frac{a_{n0}}{t-z} + \frac{a_{n1}}{t-z_r} + \ldots + \frac{a_{nn}}{(t-z_r)^n},$$

we find easily that $a_{n0} = -a_{n1} = a_n/(z-z_r)^n$. Thus, if z is at first outside C,

$$\int_C a_n\,dt/(t-z)(t-z_r)^n = -2\pi i a_n/(z-z_r)^n$$

for each pole inside C, and is zero for each pole outside C. That is, if the sum of the principal parts of $f(z)$ at the poles inside C is denoted by $f_c(z)$, $\int f(t)\,dt/(t-z) = -2\pi i f_c(z)$. Completing the C of equation (8) by a small semicircle inside the broken contour, we have

$$Hf(z) - f(z) = -2f_c(z),$$

i.e.

$$Hf(z) = f(z) - 2f_c(z). \tag{9}$$

$Hf_c(z)$ is equal to $-f_c(z)$, so that

$$H^2 f(z) = f(z). \tag{10}$$

If H^2 is a real operator, and the real or imaginary part of $f(z)$ is discontinuous at a point z, the right-hand side of (10) is replaced by $\frac{1}{2}\{f(z+0)+f(z-0)\}$, where $z+0$ and $z-0$ are indefinitely near to z on either side. Let $f(z)$, for example, be $\log\{(z+a)/(z-a)\}$, where a is real. Then C may be taken to be a line parallel to the real axis through the point z, completed at infinity by a large semicircle on the same side as z. The integral round the semicircle is zero. Thus $Hf(z) = H^2 f(z) = f(z)$; and the smaller the imaginary part of z, the more quickly the imaginary part of $f(z)$ changes near $x = \pm a$, the imaginary parts of $H^2 f(z)$ at $x = \pm a$ being those of $\frac{1}{2}\{f(x+0)+f(x-0)\}$.

As an example of equation (9), take C to be a circle of radius $|z|$ with the origin as centre. Write $t = z\exp(-i\theta)$ when $-\pi < \theta < 0$ and $z\exp i\theta$ when $0 < \theta < \pi$. Thus, if $f(z)$ has only poles inside the contour and none on it,

$$\int_0^\pi \left[\frac{f\{z\exp(-i\theta)\}}{1-\exp i\theta} + \frac{f(z\exp i\theta)}{1-\exp(-i\theta)}\right] d\theta = \pi\{f(z)-2f_c(z)\}.$$

If A and B are two circuits of which A encloses B,

$$\int_B \frac{f(t)\,dt}{t(1-t/z)} = -\int_B \frac{f(t)\,dt}{t-z} + \int_B \frac{f(t)\,dt}{t},$$

so that, if the point z lies between A and B,

$$\int_A \frac{f(t)\,dt}{t(1-z/t)} + \int_B \frac{f(t)\,dt}{t(1-t/z)} - \int_B \frac{f(t)\,dt}{t} = 2\pi i\{f(z)-f_{AB}(z)\}, \qquad (11)$$

where $f_{AB}(z)$ is the sum of the principal parts of $f(z)$ at its singularities between A and B, supposed all poles. Equation (2) follows if $f(z)$ has no singularities in the annulus.

It is sometimes helpful to regard a singularity as a centre of force. Let a uniformly electrified line, of charge e per unit length, cut the complex plane at right angles at a point t. If we call 'electric intensity' what is really its conjugate complex, the intensity at a point z is proportional to $e/(t-z)$. If e is complex and the factor of proportionality $1/2\pi i$, we have a function with a simple pole. A pole of higher order is an aggregate of charges indefinitely close together, of which the residue is the total charge. The idea of clustered charges is fundamental in Kelvin's theory of spherical harmonics. The integral

$$\frac{1}{2\pi i}\int_C \frac{f(t)\,dt}{t-z},$$

in which $f(t)$ need not be analytic nor C closed, is the electric intensity of a charged cylinder (not in general a conductor) cutting the complex plane in C, and $f(t)$ may be thought of with a certain licence as a surface density. There is a discontinuity of $\frac{1}{2}\{f(t+0)+f(t-0)\}$ when the point z crosses C, superposed on an infinity if $f(t)$ is discontinuous.

Mr. D. S. Jones has given me much help in discussing the subject of this paper, and by references to the literature.

PARAMETRIC SURFACES

III. THE PROBLEM OF GEOCZE

By E. R. REIFENBERG (*Cambridge*)

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1. THROUGHOUT this paper I shall use the notation and results of my two previous papers under the same general title (**1**), (**2**). Suppose that S is a parametric surface and that A is the class of sequences $\{S_n\}$ of polyhedra inscribed in S with their boundaries inscribed in the boundary of S, and tending to S in the sense of Frechet. I then define $L_1(S)$ by

$$L_1(S) = \inf_{A} \overline{\lim_{n}} \Lambda^2 S_n. \tag{1}$$

The problem of this paper is to show that†

$$L_1(S) = L(S). \tag{2}$$

For simplicity of exposition I shall prove the result for surfaces homeomorphic to a disk: the general result will follow at once from cyclic element theory. Since in any case

$$L(S) \leqslant L_1(S), \tag{3}$$

I shall suppose that $L(S) < \infty$.

LEMMA 1. *If $S(P_i, \delta)$ $(i = 1, 2, \ldots)$ is any set of spheres, of radius δ, such that every sphere of the set meets $S(P_1, \delta)$ and*

$$P_i P_j \geqslant \tfrac{1}{2}\delta \quad \text{for } i \neq j, \tag{4}$$

then there will be at most 729 *spheres in the set.*

The spheres $S(P_i, \frac{1}{4}\delta)$ are disjoint and contained in $S(P_1, \frac{9}{4}\delta)$, from which the result follows.

LEMMA 2. *Any set of spheres $S(P_i, \delta)$ of radius δ such that $P_i P_j \geqslant \frac{1}{2}\delta$ for $i \neq j$ can be divided into* 730 *subsets of disjoint spheres.*

The proof is identical with that of Lemma 1 of (**3**).

† This is the problem of Geocze. I find that shortly after I wrote this paper L. Césari published ['Sul Problema di Geocze', *Rivista Mat. Univ. Parma*, 1 (1950), 207–27] a solution in the following form: For any surface S there exists a representation $S = \phi(H)$ such that there exists a sequence ψ_n of functions quasi-linear on H, where $\psi_n \to \phi$ on H and $\Lambda^2\psi_n(H) \to L(S)$ while the polyhedra $\psi_n(H)$ are inscribed in S; in Césari's solution, however, the boundaries of the polyhedra $\psi_n(H)$ are not required to be inscribed in that of S. L. Césari's proof is of a completely different nature from my own.

2. Take $\epsilon > 0$. Suppose T to be the set of points (P, Q) of S^0 such that for some r as small as we please†

$$L\phi K(Q,r) \geqslant \pi r^2(1-\epsilon) \tag{5}$$

and

$$\Lambda\phi l(Q,r) < 2\pi r(1+\epsilon). \tag{6}$$

Then, by Lemma 16 and Theorem 1 of (**2**),

$$\Lambda^2 T = L(S). \tag{7}$$

A disjoint subset U of the spheres $S(P,r)$, where $P = \phi(Q)$ and (5) and (6) are satisfied, will cover Λ^2 almost all of $|T|$ by Vitali's general covering principle. Then, provided that all the r are small,

$$\tfrac{1}{4}\pi \sum_U (2r)^2 > \tfrac{3}{4}\Lambda^2|T| \tag{8}$$

and hence, by (5),

$$\sum_U L\phi K(Q,r) > \tfrac{1}{2}\Lambda^2|T|. \tag{9}$$

The curves $\phi l(Q,r)$ lie on disjoint spheres, and hence any two of the domains $K(Q,r)$ are either disjoint or one contains the other. Hence we can select a finite disjoint subset Ξ_1, of the domains $K(Q,r)$ of U, such that

$$\sum_{\Xi_1} L\phi K(Q,r) > \tfrac{1}{2}\Lambda^2|T|. \tag{10}$$

By applying the same argument to

$$S^0 - \sum_{\Xi_1} \phi\overline{K(Q,r)},$$

which is a finitely connected Jordan domain on S^0, we shall obtain a finite disjoint set Ξ_2 of domains $K(Q,r)$ contained in $H^0 - \sum_{\Xi_1} \overline{K(Q,r)}$ and such that

$$\sum_{\Xi_2} L\phi K(Q,r) > \tfrac{1}{2}\Lambda^2\Big|T - \sum_{\Xi_1} \phi\overline{K(Q,r)}\Big|, \tag{11}$$

and hence, by (6),

$$\sum_{\Xi_2} L\phi K(Q,r) > \tfrac{1}{2}\Lambda^2\Big|T - \sum_{\Xi_1} \phi K(Q,r)\Big|. \tag{12}$$

Similarly we can construct Ξ_n so that

$$\sum_{\Xi_n} L\phi K(Q,r) > \tfrac{1}{2}\Lambda^2\Big|T - \sum_{k=1}^{n-1}\sum_{\Xi_k} \phi K(Q,r)\Big|. \tag{13}$$

Since all the domains of $\sum_1^n \Xi_k$ are disjoint and $L(S) < \infty$, we have that

$$\sum_{\Xi_n} L\phi K(Q,r) \to 0 \quad \text{as } n \to \infty, \tag{14}$$

so that

$$\Lambda^2\Big|T - \sum_{k=1}^{\infty}\sum_{\Xi_k} \phi K(Q,r)\Big| = 0: \tag{15}$$

† $\phi K(Q,r)$ is the smallest simply-connected Jordan domain on S containing (P,Q) whose boundary lies on the sphere $S(P,r)$; $\phi l(Q,r)$ is the boundary of $\phi K(Q,r)$.

that is
$$\Lambda^2\Big\{T-\sum_{k=1}^{\infty}\sum_{\Xi_k}\phi K(Q,r)\Big\}=0. \tag{16}$$

Thus, by (7) and Theorem 6 of (**1**),

$$\sum_{k=1}^{\infty}\sum_{\Xi_k}L\phi K(Q,r)=L(S). \tag{17}$$

We may therefore select a finite set Ξ of domains $K(Q,r)$ such that

$$L\phi\Big\{H-\sum_{\Xi}\overline{K(Q,r)}\Big\}<\frac{\epsilon}{25{,}000}. \tag{18}$$

$H^0-\sum_{\Xi}\overline{K(Q,r)}$ is a finitely connected Jordan domain. Write

$$R'=H^0-\sum_{\Xi}\overline{K(Q,r)}$$

and let γ be the boundary of R'. Then γ consists of the boundary of H and the curves $l(Q,r)$ of Ξ.

3. The next stage in the proof consists of constructing a *finite* set Γ of *non-overlapping simply-connected Jordan* domains $G(i)\subset R'$ such that, if $g(i)$ is the part of the boundary of $G(i)$ contained in R', then $g(i)$ is *connected* and

$$\sum_{\Gamma}\overline{G(i)}=R' \tag{19}$$

while
$$\sum_{\Gamma}\{\Lambda\phi g(i)\}\{d(\phi g(i))\}<2\epsilon. \tag{20}$$

Suppose that R'' is a domain congruent and coincident with R and that R' and R'' are joined at their boundary to form a single set R. A point of R' and a point of R'' can only be joined by an arc *via* the common boundary γ. A point x' of R' and a point x'' of R'' are said to *correspond* if x' and x'' are coincident when regarded as points of the plane. Two sets $X'\subset\bar{R}'$ and $X''\subset\bar{R}''$ are said to *correspond* if, when regarded as point sets in the plane, they are identical. A set $X\subset R$ is called *self-corresponding* if $X\bar{R}'$ and $X\bar{R}''$ correspond. It is clear that a *self-corresponding simply-connected domain* which meets only *one component* of γ will meet it in a *connected set* if at all. Define ϕ_0 on R so that $\phi=\phi_0$ on $\bar{R}'$ and $\phi_0(x')=\phi_0(x'')$ if x' and x'' correspond. $S_0=\phi_0(R)$ is a parametric surface homeomorphic to R.

Clearly
$$LS_0\geqslant 2L\phi(R). \tag{21}$$

On the other hand, by taking any polyhedron, approximating to $\phi(R)$ twice we see that

$$L(S_0)\leqslant 2L\phi(R), \tag{22}$$

so that
$$L(S_0)=2L\phi(R). \tag{23}$$

The boundary γ consists of a finite number of disjoint simple closed Jordan curves on R. Let the minimum distance between any pair of them be a_1, and the minimum of their diameters a_2. Take $\eta > 0$ and suppose that

$$100\eta < \min(a_1, a_2, \epsilon);$$

ϕ_0 is bicontinuous, since $S = \phi(H)$ is homeomorphic to a disk. Hence there exists $\delta > 0$ such that, if C is any connected set in R such that

$$d\{\phi_0(C)\} < 8\delta, \quad \text{then } d\{C\} < \eta. \tag{24}$$

If we have any non-simply connected continuum X in R of diameter less than 10η, it will meet at most one of the components of γ. If X does not meet γ, then by the choice of η all of γ lies in a single component of $R-X$, which we call the *exterior complementary domain*. If X meets one component of γ, then the exterior complementary domain is the component of $R-X$ which contains the rest of γ. The other components of $R-X$ are called *interior complementary domains*.

With these definitions the results of (**1**) and (**2**) will apply to $S_0 = \phi_0(R)$ provided that all sets considered are small because S_0 is locally homeomorphic to a disk.

If X is any connected set in R, I write $\psi(X)$ for the set of interior points of the simply-connected closed set obtained from $\bar{X}$ by adding to it its interior complementary domains.

Consider the sphere $S(P, h)$. Let $\{D_1(i, P, h)\}$ be the set of components of $\phi_0^{-1}S^0(P, h)$, where i runs over a class of suffixes $\Theta(P, h)$. Let

$$D(i, P, h) = \psi\{D_1(i, P, h)\}$$

and suppose that $l(i, P, h)$ is the boundary of $D(i, P, h)$. Then, by equation (50) of (**2**),†

$$\int_0^r \sum_{\Theta(P,h)} \Lambda\phi_0 l(i, P, h)\, dh \leqslant L^*S(P, r). \tag{25}$$

Let P_1 be any point on $|S_0|$; take a point P_2 on $|S_0|$ not inside $S(P_1, \frac{1}{2}\delta)$. Then P_3 is not inside either of $S(P_1, \frac{1}{2}\delta)$ or $S(P_2, \frac{1}{2}\delta)$ and so on. Since δ is fixed and $|S_0|$ is bounded, this process must terminate. We obtain a set U of points P_j such that

$$\sum_U S^0(P_j, \tfrac{1}{2}\delta) \supset |S_0|, \tag{26}$$

and, if $j \neq j'$,

$$D_{\text{Euc}}(P_j, P_{j'}) > \tfrac{1}{2}\delta. \tag{27}$$

† L^* was defined in (**2**): it is a Carathéodory outer measure on the surface equal to the Lebesgue area on any finitely connected Jordan domain. $L^*S(P, r)$ is the L^* measure of those points of the surface inside or on $S(P, r)$.

By Lemma 2 we can divide the spheres $S(P_j, \delta)$ into 730 subsets of disjoint spheres and so, by (18) and (23),

$$\sum_U L^* S(P_j, \delta) \leqslant 730 L(S_0) \leqslant \frac{1460}{25000}\epsilon < \frac{\epsilon}{16}. \tag{28}$$

For each $P_j \in U$ there will, by (25), exist r_j such that

$$\tfrac{1}{2}\delta < r_j < \delta \tag{29}$$

and

$$\sum_{\Theta(P_j, r_j)} \Lambda\phi_0 l(i, P_j, r_j) \leqslant \frac{2L^* S(P_j, \delta)}{\delta}. \tag{30}$$

By (28) and (30),

$$\sum_{j \in U} \sum_{\Theta(P_j, r_j)} \Lambda\phi_0 l(i, P_j, r_j) < \frac{\epsilon}{8\delta}. \tag{31}$$

On the other hand, by (26),

$$\sum_{j \in U} \sum_{\Theta(P_j, r_j)} \overline{D(i, P_j, r_j)} = R \tag{32}$$

and, for $j \in U$ and $i \in \Theta(P_j, r_j)$,

$$d\{\phi_0 l(i, P_j, r_j)\} < 2\delta. \tag{33}$$

$D(i, P_j, r_j)$ is a simply-connected Jordan domain by its construction and (31). The domains $D(i, P_j, r_j)$ may overlap.

We may omit from $\sum_U \Theta(P_j, r_j)$ all those suffixes i such that the domain $D(i, P_j, r_j)$ is contained in some other domain $D(i', P_{j'}, r_{j'})$, where $j' \in U$ and $i' \in \Theta(P_{j'}, r_{j'})$, without affecting (31)–(33).

Let $\{D(i)\}$, where i runs over a class of suffixes Θ, be the components of

$$R - \sum_U \sum_{\Theta(P_j, r_j)} l(i, P_j, r_j).$$

Let $\lambda(i)$ be the boundary of $D(i)$, $i \in \Theta$. Since no domain $D(i, P_j, r_j)$ is contained in any other, $D(i)$ will be a simply-connected Jordan domain

$$\sum_\Theta \overline{D(i)} = R. \tag{34}$$

By (31),

$$\sum_\Theta \Lambda\phi_0 \lambda(i) < \frac{\epsilon}{4\delta}. \tag{35}$$

$D(i)$ $(i \in \Theta)$ will be contained in some domain $D(i_0, P_{j_0}, r_{j_0})$, where $j_0 \in U$ and $i_0 \in \Theta(P_{j_0}, r_{j_0})$. Suppose that $\lambda(i)$ meets $l(i, P_j, r_j)$, where $j \in U$ and $i \in \Theta(P_j, r_j)$. Then, since $D(i, P_j, r_j)$ is not contained in $D(i_0, P_{j_0}, r_{j_0})$, $l(i, P_j, r_j)$ will meet $l(i_0, P_{j_0}, r_{j_0})$, and hence

$$d\{\phi_0 \lambda(i)\} \leqslant d\{\phi_0 l(i_0, P_{j_0}, r_{j_0})\} + \max d\{\phi_0 l(i, P_j, r_j)\}.$$

Hence, by (33),

$$d\{\phi_0 \lambda(i)\} \leqslant 6\delta. \tag{36}$$

By (34)–(36), if Θ were a finite set and no $\overline{D(i)}$ ($i \in \Theta$) were to meet γ in a disconnected set, then the domains $R'D(i)$ would form a set Γ satisfying (19) and (20) such as we are trying to construct. As it is, we must modify the set of $D(i)$.

To each $i \in \Theta$ there will correspond an $i' \in \Theta$ such that $D(i)$ and $D(i')$ *correspond*. It is possible that $i = i'$, in which case $D(i)$ is *self-corresponding*. This is true in virtue of the symmetry of the above construction as between R' and R''.

Let Θ_0 be a finite subset of Θ such that, if $i \in \Theta_0$, then so does the corresponding i' and

$$\sum_{\Theta - \Theta_0} \Lambda \phi_0 \lambda(i) < \delta. \tag{37}$$

Let $\{\Delta(i)\}$, where i runs over a class of suffixes Θ_1, be the components of $R - \sum_{\Theta_0} \overline{D(i)} - \gamma$. To each $i \in \Theta_1$ there will correspond $i' \in \Theta_1$ such that $\Delta(i)$ and $\Delta(i')$ *correspond*.

Each domain $\Delta(i)$ ($i \in \Theta_1$) will have a common boundary arc with at least one domain $D(i)$ ($i \in \Theta_0$); take one such domain and call it the domain $D(i)$ *associated* with $\Delta(i)$. We may suppose that, if $\Delta(i)$ and $\Delta(i')$ correspond, then so will the associated domains.† For any $j \in \Theta_0$, let Γ_j be the class of $i \in \Theta_1$ such that $D(j)$ is associated with $\Delta(i)$. For $j \in \Theta_0$ let

$$D^*(j) = \Big\{\overline{D(j)} + \sum_{\Gamma_j} \overline{\Delta(i)}\Big\}^0.$$

Clearly, if $D(j)$ and $D(j')$ correspond, then so will $D^*(j)$ and $D^*(j')$; let

$$G^*(j) = \psi(\overline{D^*(j)} + \overline{D^*(j')}).$$

$G^*(j)$ will consist either of two corresponding simply-connected Jordan domains whose boundaries have at most a single common point or else of a single self-corresponding simply-connected Jordan domain. By the definition of the operator ψ and since the domains $D^*(j)$ are disjoint if $j_1 \neq j_2$, then $G^*(j_1)$ and $G^*(j_2)$ are either disjoint or one contains the other. Remove all those $j_1 \in \Theta_0$ such that $G^*(j_1)$ is contained in some other domain $G^*(j_2)$ ($j_2 \in \Theta_0$). Let the remaining set of suffixes be Γ. Write, for $j \in \Gamma$,

$$G(j) = R'G^*(j). \tag{38}$$

Let $g(j)$ be the part of the boundary of $G(j)$ in R'. Since the part of the boundary of $\Delta(i)$ ($i \in \Theta_1$) contained in R' is part of $\sum_{\Theta - \Theta_0} \lambda(i)$, we have, by (36) and (37),

$$d\{\phi g(j)\} < 8\delta, \tag{39}$$

† The domain associated with $\Delta(i)$ may coincide with that associated with $\Delta(i')$, and in that case it is self-corresponding.

and so, by (35), (20) will hold. Also, since $\overline{G^*(j)}$ either meets γ in a single point or else is self-corresponding, $g(j)$ will be connected, and the construction of Γ is complete.

4. Take $j \neq j' \in \Gamma$. Consider $g(j)-g(j)g(j')$. This will consist of a set of open arcs on $g(j)$. These arcs will be finite in number, for each such arc will contain boundary points of a domain of Γ other than $G(j)$ or $G(j')$, and no domain of Γ other than these can have boundary points on more than one arc of $g(j)-g(j)g(j')$. Let $\nabla(j,j')$ be the set of end-points of such arcs plus the end-points of $g(j)$ in the case where $\overline{G(j)}$ meets γ.

Let
$$\nabla = \sum_{j \in \Gamma} \sum_{j \neq j'} \nabla(j,j'). \tag{40}$$

Then ∇ is a finite set of points.

Let P', P'' be two consecutive points of ∇ on $G(j)$, so that one of the arcs $p'p''$ of $g(j)$ contains no points of ∇. Let C be this arc. Define ϕ^* as a continuous function on C such that $\phi^*(C)$ is the straight line joining $\phi(P')$ and $\phi(P'')$.

If any curve $g(j')$ contains points of C, then it will contain the whole of C by the definition of ∇, and so ϕ^* is independent of j. We can thus define ϕ^* uniquely on $\sum_{\Gamma} g(j)$.

$\phi^* g(j)$ will be a polygon $\pi(j)$ whose vertices lie on $\phi(R')$ except that, when $\overline{G(j)}$ meets γ, $\phi^* g(j)$ will be a polygonal arc whose end-points are on $\phi(\gamma)$. Suppose that the vertices of $\pi(j)$ are $P_1 P_2 \ldots P_n$ taken in order, and that, when $\overline{G(j)}$ meets γ, so that $\pi(j)$ is a polygonal arc, then P_1 and P_n are its end-points. Let $\varpi(j)$ be the polyhedral surface consisting of the triangles $P_1 P_k P_{k+1}$ $(2 \leqslant k \leqslant n-1)$. Then, since $G(j)$ is a simply-connected Jordan domain, there exists a continuous function ϕ_j^* defined on $\overline{G(j)}$ and equal to ϕ^* on $g(j)$ such that

$$\phi_j^*\{\overline{G(j)}\} = \varpi(j). \tag{41}$$

Define ϕ^* on the whole of $\bar{R}'$ by writing $\phi^* = \phi_j^*$ on $\overline{G(j)}$. Since Γ is a finite set, ϕ^* is continuous.

By (39),
$$d\{\pi(j)\} \leqslant d\{\phi g(j)\} \leqslant 8\delta. \tag{42}$$

Also, by (20),
$$\Lambda^2 \phi^*(R) \leqslant \sum_{\Gamma} \tfrac{1}{2} \Lambda \pi(j) d\{\pi(j)\} \leqslant \sum_{\Gamma} \tfrac{1}{2} \Lambda \phi g(j) d\{\phi g(j)\} \leqslant \epsilon. \tag{43}$$

Suppose now that $K(Q,r)$ is a domain of Ξ. Then $\phi^* l(Q,r)$ is a polygon inscribed in $\phi l(Q,r)$. Let $P_1 \ldots P_n$ be this polygon. Let $\varpi(Q,r)$ be the polyhedral surface consisting of the triangles $P_1 P_k P_{k+1}$ $(2 \leqslant k \leqslant n-1)$.

Then, by (5) and (6), since $\varpi(Q,r)$ may be developed,

$$\Lambda^2\varpi(Q,r) \leqslant (1+\epsilon)^2\pi r^2 \leqslant \frac{(1+\epsilon)^2}{1-\epsilon} L\phi K(Q,r). \tag{44}$$

We can define ϕ^* so that $\phi^*K(Q,r) = \varpi(Q,r)$. Then ϕ^* will be continuous on the whole of H, and $\phi^*(H)$ is a polyhedral surface while, by (43) and (44),

$$\Lambda^2\phi^*(H) \leqslant \frac{(1+\epsilon)^2}{1-\epsilon} L(S)+\epsilon. \tag{45}$$

On the other hand,

$$d\{\varpi(j)\} \leqslant d\{\pi(j)\} \leqslant d\{\phi g(j)\},$$

and
$$d\{\varpi(Q,r)\} \leqslant d\{\phi^* l(Q,r)\} \leqslant d\{\phi l(Q,r)\};$$

so for any x
$$|\phi(x)-\phi^*(x)| \leqslant 2\max\{|\phi(x)-\phi(x')|\}$$

where the maximum is taken over all pairs x, x' belonging to the same $\overline{G(j)}$ or $\overline{K(Q,r)}$. But, by (24) and (39), $d\{G(j)\} < \epsilon$ and we may suppose that all the r were chosen so small that $d\{K(Q,r)\} < \epsilon$ also.

If ϵ is small, we then have by the continuity of ϕ that $\phi^*(H)$ is near $\phi(H)$; and hence from (45) we obtain

$$L_1(S) \leqslant L(S). \tag{46}$$

This together with (3) establishes the desired result.

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AN INTERPOLATORY FUNCTION ANALOGOUS TO THE CARDINAL FUNCTION

By R. C. T. SMITH (*Armidale, N.S.W.*)

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IF the series
$$C(z) = \frac{\sin \pi z}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n a_n}{z-n}, \tag{1}$$
converges for any z not an integer, then it converges uniformly for z in any fixed finite region and $C(z)$ is an integral function.† $C(z)$ is known as the *cardinal function*. It is an interpolatory function taking the values a_n at the integers and it has the following important property of consistency.‡,§ If $C(z)$ is defined by the series (1), then under suitable conditions on the a_n,
$$C(z) = \frac{\sin \pi(z-\lambda)}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n C(n+\lambda)}{z-n-\lambda}.$$

It is perhaps of interest to consider a similar function for interpolation at the non-negative integers and we easily see that
$$G(z) = -\frac{1}{\Gamma(-z)} \sum_{r=0}^{\infty} \frac{(-1)^r b_r}{r!\,(z-r)} \tag{2}$$
is an integral function with given values b_r at the non-negative integers provided that the series converges for some z not an integer. The convergence theory is a special case of that of (1) if we take $a_r = 0$ for $r = -1, -2, \ldots$ in (1).

I shall prove the consistency property
$$G(x) = -\frac{1}{\Gamma(-x+\lambda)} \sum_{n=0}^{\infty} \frac{(-1)^n G(n+\lambda)}{n!\,(x-n-\lambda)} \quad (\mathrm{re}\,\lambda < 0) \tag{3}$$
of the function $G(z)$ defined by the series (2) (assumed convergent) for either of the two conditions

(*a*) $\displaystyle\sum_{r=0}^{\infty} \left|\frac{b_r}{r!}\right|^p < \infty \quad (p > 1)$;

(*b*) $\displaystyle\frac{b_r}{r!} = o(r^{\delta}) \quad (\delta \text{ arbitrary} > 0)$.

† J. M. Whittaker, *Proc. Edinburgh Math. Soc.* (2) 1 (1927), 41–6.

‡ J. M. Whittaker, *Interpolatory Function Theory* (Cambridge, 1935), 68.

§ W. L. Ferrar, *Proc. Roy. Soc. Edinburgh*, 47 (1927), 230–42. This paper is referred to as 'Ferrar' in the sequel.

I shall also show the connexion between (3) and a series-inversion formula due to Linfoot and Shepherd.† Their work gives an example showing that (3) does not hold for $\mathrm{re}\,\lambda > 0$. The method used is similar to that of Ferrar and makes use of lemmas proved in his paper.

We shall need the following two results:

LEMMA 1. $\dfrac{\Gamma(z+\lambda)}{\Gamma(z)} = O(|z|^{\mathrm{re}\,\lambda})$ *as* $|z| \to \infty$, *on the square* C_N $(N = 1, 2,\ldots)$ *with corners* $(N+k)\pm(N+k)i$ $(0 < k < 1)$, *provided that neither of* $\mathrm{re}\,\lambda-k$, $\mathrm{re}\,\lambda+k$ *is an integer.*

From the asymptotic formula

$$\log\Gamma(z) = (z-\tfrac{1}{2})\log z - z + \tfrac{1}{2}\log 2\pi + o(1)$$

valid for $-\pi+\epsilon \leqslant \arg z \leqslant \pi-\epsilon$ (ϵ constant > 0), we obtain

$$\frac{\Gamma(z+\lambda)}{\Gamma(z)} \sim z^{\lambda}.$$

Now $$|z^{\lambda}| = |z|^{\mathrm{re}\,\lambda}\exp(-\arg z\,\mathrm{im}\,\lambda) = O(|z|^{\mathrm{re}\,\lambda}),$$

which proves the lemma for $-\pi+\epsilon \leqslant \arg z \leqslant \pi-\epsilon$. For

$$-\pi \leqslant \arg z < -\pi+\epsilon, \quad \pi-\epsilon < \arg z \leqslant \pi,$$

i.e. for $$-\epsilon < \arg(-z) < \epsilon,$$

we first make use of the functional equation

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z},$$

to obtain $$\frac{\Gamma(z+\lambda)}{\Gamma(z)} = \frac{\sin\pi z}{\sin\pi(z+\lambda)}\,\frac{\Gamma(1-z)}{\Gamma(1-z-\lambda)}.$$

Lemma III of Ferrar shows that the upper bound of $\sin\pi z/\sin\pi(z+\lambda)$ on C_N remains finite as $N \to \infty$ provided that neither of $\mathrm{re}\,\lambda-k$, $\mathrm{re}\,\lambda+k$ is an integer. Now

$$\frac{\Gamma(1-z)}{\Gamma(1-z-\lambda)} \sim (1-z)^{\lambda}$$

for $-\epsilon < \arg(-z) < \epsilon$ and combining these two results proves the lemma for the values of $\arg z$ not already considered.

† E. H. Linfoot and W. M. Shepherd, *Quart. J. Math.* (Oxford), 10 (1939), 84–98.

LEMMA 2. $\sum_{r=0}^{\infty} \frac{\alpha_r}{z-r} = o(|z|^{\delta'})$ *on* C_N (δ' *arbitrary* > 0) *for either of the two conditions*

(*a*) $\sum_{r=0}^{\infty} |\alpha_r|^p < \infty \quad (p > 1)$;

(*b*) $\sum_{r=0}^{\infty} \frac{\alpha_r}{z_0-r}$ *convergent for some* z_0 *not an integer;*

$$\alpha_r = o(r^{\delta}) \quad (\delta \text{ arbitrary} > 0).$$

(*a*) Lemma II of Ferrar shows that in this case

$$\sum_{r=0}^{\infty} \frac{\alpha_r}{z-r} = o(1).$$

(*b*) Suppose that z is on the line joining $(N+k)\pm(N+k)i$ and z_0 is contained in the square C_N. For convenience in writing we suppose N to be even. (If N is odd $\frac{1}{2}N$ should be replaced by its integral part $[\frac{1}{2}N]$.) We write

$$\sum_{r=0}^{\infty} \frac{\alpha_r}{z-r} = \sum_{0}^{\frac{1}{2}N} + \sum_{\frac{1}{2}N+1}^{N} + \sum_{N+1}^{\frac{3}{2}N} + \sum_{\frac{3}{2}N+1}^{\infty}.$$

For the first term, Abel's lemma for complex series shows that

$$\left|\sum_{r=0}^{\frac{1}{2}N} \frac{\alpha_r}{z-r}\right| = \left|\sum_{r=0}^{\frac{1}{2}N} \frac{\alpha_r}{z_0-r}\,\frac{z_0-r}{z-r}\right| < H\left\{\sum_{0}^{\frac{1}{2}N-1} \left|\frac{z_0-r}{z-r} - \frac{z_0-r-1}{z-r-1}\right| + \left|\frac{z_0-\frac{1}{2}N}{z-\frac{1}{2}N}\right|\right\}$$

$$= H\left\{\sum_{0}^{\frac{1}{2}N-1} \left|\frac{z-z_0}{(z-r)(z-r-1)}\right| + \left|\frac{z_0-\frac{1}{2}N}{z-\frac{1}{2}N}\right|\right\},$$

where H is the upper bound of

$$\left|\sum_{m+1}^{m+p} \frac{\alpha_r}{z_0-r}\right|$$

for unrestricted m, p. Now

$$|z-z_0| < 2\sqrt{2}(N+k), \quad |z-r| \geqslant \tfrac{1}{2}N+k \quad \text{for } r \leqslant \tfrac{1}{2}N,$$

so that

$$\left|\frac{z-z_0}{(z-r)(z-r-1)}\right| < \frac{2\sqrt{2}(N+k)}{(\frac{1}{2}N+k)^2},$$

and

$$\left|\sum_{0}^{\frac{1}{2}N} \frac{\alpha_r}{z-r}\right| < H\left\{\frac{\sqrt{2}\,N(N+k)}{(\frac{1}{2}N+k)^2} + \frac{2\sqrt{2}(N+k)}{\frac{1}{2}N+k}\right\} = O(1).$$

Similarly

$$\left|\sum_{\frac{1}{2}N+1}^{\infty} \frac{\alpha_r}{z-r}\right| < H\left\{\sum_{\frac{1}{2}N+1}^{\infty} \left|\frac{z-z_0}{(z-r)(z-r-1)}\right| + \lim_{n\to\infty}\left|\frac{z_0-n}{z-n}\right|\right\} = O(1).$$

The sums $\sum_{\frac{1}{2}N+1}^{N}$, $\sum_{N+1}^{\frac{1}{2}N}$ determine the order of magnitude of the function. Consider

$$\sum_{\frac{1}{2}N+1}^{N} \frac{\alpha_r}{z-r},$$

and let A_n be the greatest of $|\alpha_{\frac{1}{2}N+1}|,\ldots, |\alpha_N|$. Then, since

$$|z-r| > N+k-r,$$

$$\left|\sum_{\frac{1}{2}N+1}^{N} \frac{\alpha_r}{z-r}\right| < A_N \sum_{\frac{1}{2}N+1}^{N} \frac{1}{N+k-r} = O(A_N \log N).$$

If $\alpha_r = o(r^\delta)$, then $A_N = o(N^\delta)$, and $A_N \log N = o(N^{\delta'})$ for $\delta' > \delta$. The sum $\sum_{N+1}^{\frac{1}{2}N}$ is treated in a similar manner and gives the same result. Thus for z on the line joining $(N+k)\pm(N+k)i$

$$\sum_{0}^{\infty} \frac{\alpha_r}{z-r} = o(N^{\delta'}) = o(|z|^{\delta'}).$$

The remaining portions of C_N can be treated in a similar way.

Now consider the sequence of integrals

$$\int_{C_N} = \frac{1}{2\pi i} \int_{C_N} \frac{-\Gamma(-z+\lambda)G(z)}{z-x}\, dz \quad (N = 1, 2,\ldots), \tag{4}$$

where $G(z)$ is defined by (2), C_N is as in Lemma 1 with $\operatorname{re}\lambda - k$, $\operatorname{re}\lambda + k$ non-integral.

The residue at $z = x$ is

$$-\Gamma(-x+\lambda)G(x)$$

provided that $x \neq p+\lambda$ (p a positive integer or zero), while the residue at $z = n+\lambda$ is

$$-\frac{(-1)^n G(n+\lambda)}{n!\,(n+\lambda-x)}.$$

Thus the value of the integral (4) is

$$-\Gamma(-x+\lambda)G(x) - \sum_{n=0}^{[N+k-\lambda]} \frac{(-1)^n G(n+\lambda)}{n!\,(n+\lambda-x)}. \tag{5}$$

In Lemma 1 we assume $\operatorname{re}\lambda < 0$ and in Lemma 2 we write $\alpha_r = \dfrac{(-1)^r b_r}{r!}$ and take $\delta' = -\operatorname{re}\lambda$. This gives $\Gamma(-z+\lambda)G(z) = o(1)$ on C_N and therefore

$$\int_{C_N} = \int_{C_N} \frac{o(1)}{z-x}\, dz \to 0 \tag{6}$$

as $N \to \infty$ uniformly for x in any finite region. From (5)

$$G(x) = -\frac{1}{\Gamma(-x+\lambda)} \sum_{n=0}^{\infty} \frac{(-1)^n G(n+\lambda)}{n!\,(x-n-\lambda)}, \tag{3}$$

provided that x is not of the form $p+\lambda$ (p a positive integer or zero). Since the series on the right converges uniformly for x in any finite region (in particular at the points $x = \lambda, 1+\lambda, \ldots$), it is an integral function, and the two sides of (3) must be equal everywhere. The consistency property has now been proved.

I shall change the sign of λ in Linfoot and Shepherd's results to make them agree with the notation of the present paper. Linfoot and Shepherd show that the equations

$$\gamma_n = -\frac{\sin \pi\lambda}{\pi} \sum_{r=0}^{\infty} \frac{\Gamma(r-\lambda+1) A_r}{r!\,(r-n-\lambda)},$$

have the unique solution

$$A_r = \frac{\sin \pi\lambda}{\pi} \sum_{n=0}^{\infty} \frac{\Gamma(n+\lambda+1)\gamma_n}{n!\,(\lambda+n-r)},$$

under the conditions $\lambda < 0$, $\sum_{n=1}^{\infty} \frac{|\gamma_n| \log n}{n} < \infty$.

If $\lambda < -1$, there are additional 'consistency conditions' on the γ_n. It is shown that

$$\sum_{r=1}^{\infty} \frac{1}{r} \left| \frac{\Gamma(r-\lambda+1)}{r!} A_r \right| < \infty.$$

This result can be interpreted as a particular case of the consistency property. If we define

$$G(n+\lambda) = -\frac{1}{\Gamma(-n-\lambda)} \sum_{r=0}^{\infty} \frac{(-1)^r G(r)}{r!\,(n+\lambda-r)},$$

then the consistency property shows that

$$G(r) = -\frac{1}{\Gamma(-r+\lambda)} \sum_{n=0}^{\infty} \frac{(-1)^n G(n+\lambda)}{n!\,(r-n-\lambda)},$$

provided that $\lambda < 0$ and the $G(r)$ satisfy a suitable convergence condition e.g. (a) or (b) of Lemma 2 with $\alpha_r = (-1)^r G(r)/r!$. If we let

$$-\Gamma(-n-\lambda)G(n+\lambda) = \gamma_n, \qquad \Gamma(-r+\lambda)G(r) = A_r,$$

and use the functional equation $\Gamma(z)\Gamma(1-z) = \pi \operatorname{cosec} \pi z$, we obtain

Linfoot and Shepherd's formulae. Their convergence condition on the γ_n is equivalent to

$$\sum_{n=1}^{\infty} \frac{|G(n+\lambda)|}{n!}\, n^{-\lambda-1} \log n < \infty,$$

and the condition proved on the A_r to

$$\sum_{r=1}^{\infty} \frac{1}{r} \left| \frac{G(r)}{r!} \right| < \infty,$$

i.e. to absolute convergence of the series defining $G(z)$.

Linfoot and Shepherd also show that the homogeneous equations

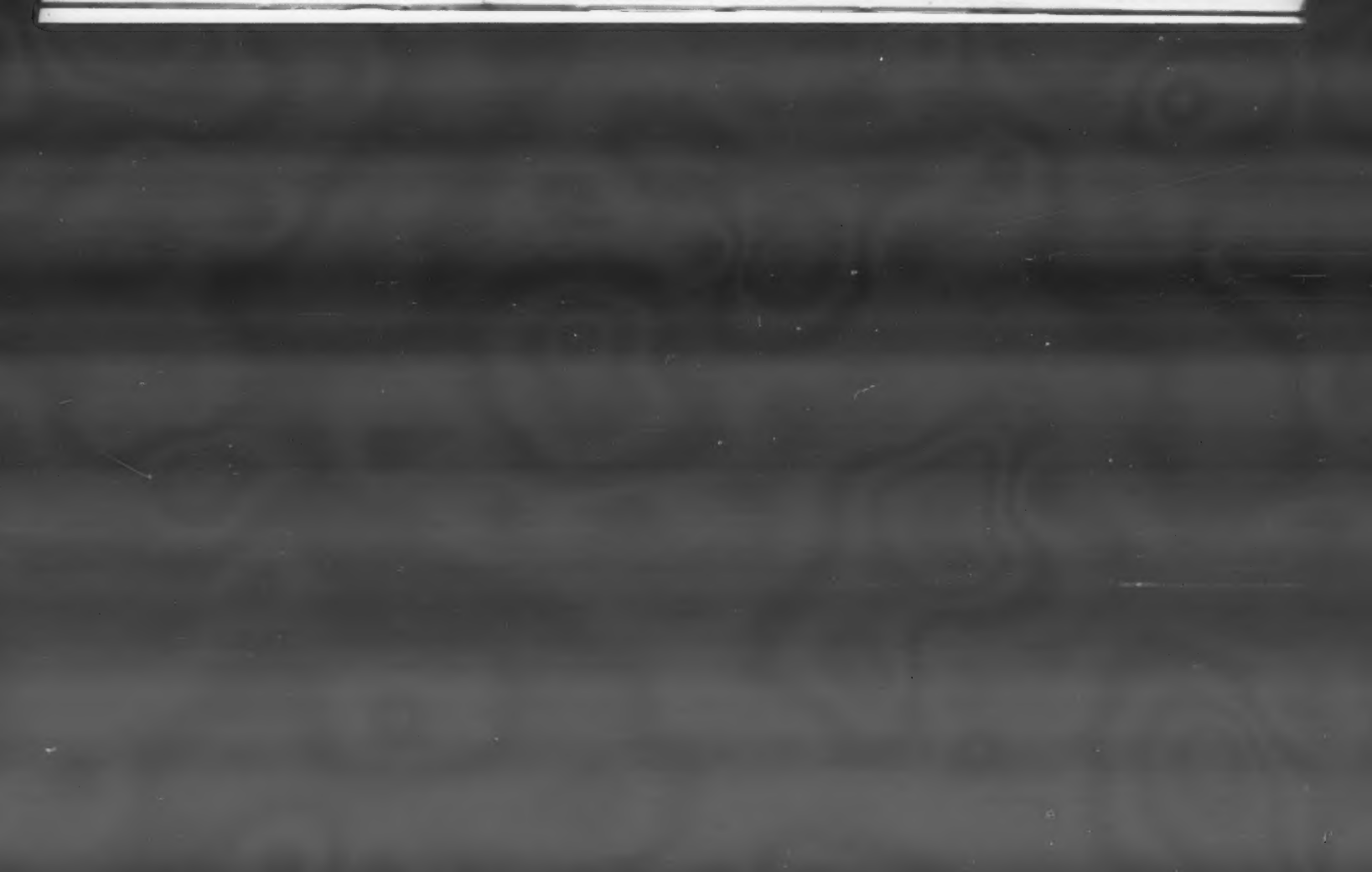
$$0 = \sum_{r=0}^{\infty} \frac{\alpha_r}{\lambda+n-r} \tag{7}$$

have solutions with not all α_r zero if $\lambda > 0$. If we write $G(r) = (-1)^r r!\, \alpha_r$, this shows that the consistency property does not hold for $\lambda > 0$ since

$$G(n+\lambda) = 0 \quad (n = 0, 1, 2, \ldots),$$

but
$$G(r) \neq -\frac{1}{\Gamma(-r+\lambda)} \sum_{n=0}^{\infty} \frac{(-1)^n G(n+\lambda)}{n!\,(r-n-\lambda)} = 0.$$

It is quite straightforward to verify that the solutions of (7) obtained by Linfoot and Shepherd for $\lambda > 0$ are still solutions if we assume λ complex, $\mathrm{re}\,\lambda > 0$. Thus the consistency property cannot hold for $\mathrm{re}\,\lambda > 0$.



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